Competitive division of a mixed manna*

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Abstract

A mixed manna contains *goods* (that everyone likes), *bads* (that everyone dislikes), as well as items that are *goods* to some agents, but *bads* or satiated to others.

If all items are goods and utility functions are *homogeneous of degree one*, concave (and monotone), the competitive division maximizes the Nash product of utilities (Gale-Eisenberg): hence it is *welfarist* (determined by the set of feasible utility profiles), unique, continuous and easy to compute.

We show that the competitive division of a mixed manna is still welfarist. If the zero utility profile is Pareto dominated, the competitive profile is strictly positive and still uniquely maximizes the product of utilities. If the zero profile is unfeasible (for instance if all items are bads), the competitive profiles are strictly negative, and are the critical points of the product of *dis*utilities on the efficiency frontier. The latter allows for multiple competitive utility profiles, from which no single-valued selection can be continuous or *Resource Monotonic.*

Thus the implementation of competitive fairness under linear preferences in interactive platforms like SPLIDDIT will be more difficult when the manna contains bads that overwhelm the goods.

1 Introduction

The literature on fair division of private commodities, with few exceptions discussed in Section 2, focuses almost exclusively on the distribution of disposable commodities, i. e., desirable *goods* like a cake (Steinhaus (1948) [38]), family heirlooms (Pratt, Zeckhauser (1990) [29]), the assets of divorcing partners (Brams, Taylor (1996) [4]), office space between co-workers, seats in overdemanded business school courses (Sönmez,Ünver (2010) [37], Budish, Cantillon (2010) [7]), computing resources in peer-to-peer platforms (Ghodsi et al. (2011) [14]), and so on. Many important fair

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division problems involve *bads* (non disposable items generating disutility): family members distribute house chores, workers divide job shifts (Budish (2011) [6]) like teaching loads, cities divide noxious facilities, managers allocate cuts within the firm, and so on. Moreover the bundle we must divide (the *manna*) often contains the two types of items: dissolving a partnership involves distributing its assets as well as its liabilities, some teachers relish certain classes that others loathe, the land to be divided may include polluted as well as desirable areas, and so on. And the manna may contain items, such as shares in risky assets, or hours of baby-sitting, over which preferences are single-peaked without being monotone, so they will not qualify as either "good" or "bad", they are "satiable" items. Of course each item may be a good to some agents, a bad to others, and satiable to yet other agents. We will speak of a *mixed manna*. Our paper is, to the best of our knowledge, the first to address the interplay of these different types of items in the fair division problem.

To see why it is genuinely more complicated to divide a mixed rather than a good or a bad manna, consider the popular fairness test of *Egalitarian Equivalence* (EE) due to Pazner and Schmeidler (1978) [27]. A division of the manna is EE if everyone is indifferent between her share and some common reference share: with mixed items this property may well be incompatible with Efficiency.[1] The news is much better for the division favored by microeconomists for over four decades (Varian (1974) [42]), the *Competitive Equilibrium with Equal Incomes* (here Competitive division, for short). Existence is guaranteed when preferences are convex, continuous, but not necessarily monotonic and possibly satiated: see e. g., Shafer and Sonnenschein (1975) [35], Mas-Colell (1982) [23]. And this division retains the key normative properties of Efficiency, No Envy, and Core stability from equal initial endowments.

A striking result by Gale, Eisenberg, and others (Gale (1960) [13], Eisenberg (1961) [12], Chipman (1974) [9], Shafer, Sonnenschein (1993) [36]) shows that in the subdomain of utilities *homogeneous of degree one* (1-homogeneous, for short), as well as concave and continuous, the competitive division of *goods* obtains by simply maximizing the product of individual utilities. This is remarkable for three reasons. First the "resourcist" concept of competitive division guided by a price balancing Walrasian demands, has an equivalent "welfarist" interpretation as the Nash bargaining solution of the feasible utility set. Second, the competitive utility profile is unique because by the latter definition it solves a strictly convex optimization program; it is also computationally easy to find and continuous with respect to the parameters of individual utilities (Vazirani (2005) [43], Megiddo, Vazirani (2007) [24]); these properties all fail under general Arrow-Debreu preferences. Finally the result is broadly applicable because empirical work relies mostly on 1-homogeneous utilities, that include additive, Cobb Douglas, CES, Leontief, and their positive linear combinations. So the Gale Eisenberg theorem is arguably the most compelling practical vindication of the competitive approach to the fair division of goods.

Main result We generalize this result to the division of a mixed manna, when utilities are concave, continuous and 1-homogeneous, but not necessarily monotonic. We show that the welfarist interpretation of the competitive division is preserved: the set of feasible utility profiles is still all we need to know to identify the competitive utility profiles (those associated with a competitive division of the items), and they are still related to the product of utilities or disutilities. A description of our main result, the Theorem in Section 4, follows.

[1] Two agents $1, 2$ share (one unit of) two items a, b, and their utilities are linear: $u_1(z_1) = z_{1a} - 2z_{1b}$; $u_2(z_2) = -2z_{2a} + z_{2b}$. The only efficient allocation gives a to 1 and b to 2. In an EE allocation (z_1, z_2) there is some $y \geq 0$ such that $u_i(z_i) = u_i(y)$ for $i = 1, 2$. This implies $u_1(z_1) + u_2(z_2) = -(y_a + y_b)$ so that z is not efficient.

We identify a partition of all division problems in three types, determined by a very simple welfarist property. Keep in mind that, by 1-homogeneity of utilities, the zero of utilities corresponds to the state of the world without any manna to divide. Call an agent "attracted" if there is a share of the manna giving her strictly positive utility, and "repulsed" if there is none, that is to say zero is her preferred share.

If it is feasible to give a positive utility to all attracted agents, and zero to all repulsed ones, we call this utility profile "positive" and speak of a "positive" problem. Problems of this type include those with *only goods*, but also all those where, intuitively, the goods can overwhelm the bads.Then the competitive utility profile is positive and maximizes the product of the attracted agents' utilities over positive profiles; like in Gale Eisenberg this utility profile is unique and easy to compute. Also, upon arrival of the manna everyone's utility increases (at least weakly).

By contrast, if the efficiency frontier contains allocations where *everyone* gets a strictly negative utility, we call the problem "negative". The division of only bads is a negative problem, but this class also includes all problems with not enough goods to overwhelm the bads. Then the competitive utility profiles are the *critical points* (for instance local maxima or minima) of the product of all **dis**utilities on the intersection of the efficiency frontier with the (strictly) negative orthant: in particular the arrival of the manna implies a (strict) utility loss for everyone.

Finally the "null" problems are those knife-edge cases where the zero utility profile is efficient: it is then the unique competitive utility profile, and the arrival of the manna has no impact on welfare.

Beyond the above similarities of the competitive approach to fair division in positive and negative problems, there are also important differences.

First, computing the efficient disutility profiles critical for the product of disutilities is no longer a convex program so we expect computational difficulties in problems with many agents and/or items. Second, negative problems routinely have multiple competitive allocations with distinct utility profiles. We show for instance that the number of competitive divisions can grow at least exponentially in the (minimum) of the number of agents and goods. Third, any single-valued selection from the competitive correspondence has unpalatable features described below.

Applications to practical fair division Our results have implications for the user-friendly internet platforms like SPLIDDIT (www.spliddit.org/) or ADJUSTED WINNER (www.nyu.edu/projects/adjustedwinner/), computing fair outcomes in a variety of problems including the division of manna. Visitors of these sites must distribute 1000 points over the different items, and these "bids" are interpreted as fixed marginal utilities for goods, or marginal disutilities for bads. Thus a participants *must* report linear preferences (additive utilities), a fairly manageable task if complementarities between items can be ignored. Thousands of visitors have used SPLIDDIT since November 2014, fully aware of the interpretation of their bids (Goldman, Procaccia (2014) [15]), is proof enough that the latter assumption is often acceptable in practical division problems.

SPLIDDIT proposes the competitive solution to divide goods, but for the division of tasks it reverts to the Egalitarian Equivalent solution mentioned in the second paragraph. That solution may generate envy in problems with three or more agents, and has several other undesirable features explained in Bogomolnaia et al.(2016) [3]. Moreover, as mentioned above the EE approach collapses for a mixed manna. So it is tempting to recommend instead the competitive approach for the division of a mixed manna.

Additive utilities are the simplest domain to which our result applies, therefore we expect multiple competitive allocations when dividing bads and more generally in negative problems.

Multiplicity is a useful qualitative insight when the interpretation of the equilibrium is descriptive, but our prescriptive approach looks for unambiguous answers to the fair division problem.

If a negative problem involves only two agents with additive utilities, there are several natural ways to single out a particular competitive utility profile. The set of Pareto optimal (dis)utility profiles is a line made of several segments, along which sit (generically) an odd number of competitive profiles, so we can for instance pick the "median" profile. In a (necessarily negative) problem with two bads the set of Pareto optimal and Envy Free allocations is similarly a line connecting several segments, and it contains (generically) an odd number of competitive allocations, so the median allocation is again a reasonable compromise. For general problems we propose a simple but not particularly compelling selection that *maximizes* the product of *dis*utilities on the strictly negative part of the efficiency frontier.

However it turns out that in negative problems *any* single valued selection from the set of efficient and non envious divisions (a much larger set than the competitive correspondence) has two undesirable features that do not affect positive problems. First, it is discontinuous in the parameters of the problem (the profile of additive utilities): Proposition 1 Section 7. Second, it violates the familiar *Resource Monotonicity* axiom (RM), requiring solidarity in individual welfares when the manna improves: if we increase the amount of a good, or decrease that of a bad, everyone should benefit at least weakly.[2] Moreover the latter impossibility result holds if we replace No Envy by the much weaker *Fair Share Guarantee* property[3]: Proposition 2 Section 7.

We conclude that the practical implementation of competitive division, on these internet platforms or elsewhere, is feasible for negative problems but not as palatable as for positive problems.

Turning bads into goods? It is is somewhat counter-intuitive (it certainly surprised us) that dividing bads competitively proves so different than dividing goods. Indeed the division of bads can be turned into that of goods in the same way as we view labor as the negative of leisure. Say that we must allocate 5 hours of the painful job a among three agents. Working 2 hours on a is the same as being exempt from a for 3 hours, so the distribution of 5 hours of a is equivalent to that of 10 hours of "a-exemption" between the three agents. Repeating this for each job, the *bad* manna (the jobs) becomes a *good* manna (the exemptions). But in the new problem no agent can eat more than 5 hours of a-exemption, and these additional capacity constraints prevent us to apply the Gale Eisenberg theorem. It is well known that capacity constraints on individual consumption enlarges the set of competitive allocations.

Contents After reviewing the literature in Section 2 and defining the model in Section 3, we state our main result in Section 4. We illustrate the multiplicity issue in a series of problems with additive utilities in Section 5; most of the examples involve two agents or two items. For negative problems with additive utilities, Section 6 defines the single valued competitive rules mentioned above, then Section 7 state two impossibility results. All substantial proofs are in Section 8.

[2]RM has been applied to many different resource allocation problems with production and/or indivisibilities. See the recent survey by Thomson (2010) [40].

[3]That is, no one is worse off than by consuming a $\frac{1}{n}$-th share of every item. Equal split plays the role of the disagreement option in bargaining.

2 Related literature

1. Steinhaus' (1948) [38] "cake-division" describes the "good" cake as a compact euclidean set, and assume linear preferences represented by atomless measures over the cake. This model contains ours (for goods) as the special case where the measures have piecewise constant densities. Segal-Halevi and Sziklai (2015) 2015 [33] show that the competitive allocations are still the Nash product maximizer, hence generalizing Eisenberg Gale. They also show that the competitive rule is Resource Monotonic. The cake division literature pays some attention to the division of a *bad* cake, to prove the existence of envy-free divisions of the cake (Su (1999) [39], Peterson, Su (2002) [28], Heydrich, van Stee (2015) [18]), or to examine how the classic algorithms by cuts and queries can or cannot be adapted to this case (Brams, Taylor (1996) [4], Robertson, Webb (1998) [30]). It does not discuss the competitive rule for a bad cake or a mixed cake. Following our work Segal-Halevi (2017) [32] discusses the existence of non envious division of a cake with good and bad parts.

2. Our paper Bogomolnaia et al. (2016) [3] (thereafter the *companion paper*) focuses on the contrast between "all goods manna" and "all bads manna" problems in the additive domain. It compares systematically the Competitive and Egalitarian Equivalent division rules, vindicating the former rule by additional independence and monotonicity properties. It contains the proof of some of the results stated here in Sections 5 and 7.

3. The recent work in computational social choice discusses extensively the fair division of goods (see the survey Brandt et al. (2016) [5]), recognizing the practical convenience of additive utilities and the conceptual advantages of the competitive solution in that domain (see Moulin (2003) [25], Vazirani (2005) [43]). For instance Megiddo and Vazirani (2007) [24] show that the competitive utility profile depends continuously upon the rates of substitution and the total endowment; Jain and Vazirani (2010) [20] that it can be computed in time polynomial in the dimension $n+m$ of the problem (number of agents and of goods).

4. The fair division of *indivisible goods* with additive utilities is a much studied variant of the standard model. The maximization of the Nash product loses its competitive interpretation and becomes hard to compute (Lee (2015) [21]), however it is envy-free "up to at most one object" (Caragiannis et al. (2016) [8]) and can be efficiently approximated for many utility domains (Cole, Gkatzelis (2015) [11], Anari et al. (2016) [1], Anari et al. (2016) [2], Cole et al. (2016) [10]). Also Budish (2011) [6] approximates the competitive allocation in problems with a large number of copies of several good-types by allowing some flexibility in the number of available copies.

5. The probabilistic assignment of goods with von Neuman Morgenstern utilities is another fair division problem with linear and satiated preferences where Hylland and Zeckhauser (1979) [19] and the subsequent literature recommend (a version of) the competitive rule: e. g., He et al. (2015) [16]. That rule is no longer related to the maximization of the product of utilities.

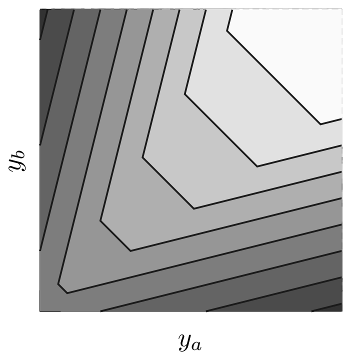
6. The purely welfarist axiomatic discussion of non convex bargaining problems identifies the set of critical points of the Nash product among efficient utility profiles as a natural generalization of the Nash solution: Herrero (1989) [17], Serrano, Shimomura (1998) [34]. This solution stands out also in the rationing model of Mariotti and Villar (2005) [22] where we divide utility losses instead of gains. The latter is closer in spirit to our results for the division of bads.

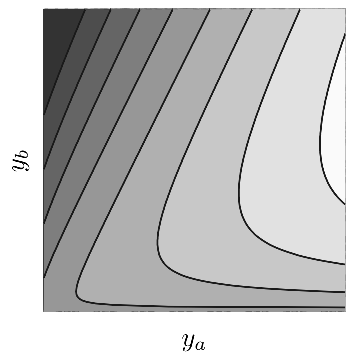
3 The model

Notation: we write $\mathbb{R}^X_+, \mathbb{R}^X_-, \mathbb{R}^X_{++}, \mathbb{R}^X_=$, for, respectively, the non negative, non positive, strictly positive and strictly negative vectors in $\mathbb{R}^X$.

The set of agents is N, that of items is A; both are finite. The domain $\mathcal{H}(A)$ consists of all preferences on $\mathbb{R}_+^A$ represented by a real-valued utility function v on $\mathbb{R}_+^A$ that is concave, continuous, and 1-homogeneous: $v(\lambda y) = \lambda v(y)$ for all $\lambda \geq 0, y \in \mathbb{R}_+^A$. It is easily checked that if two such utility functions represent the same preference, they differ by a positive multiplicative constant. All our definitions and results are purely ordinal, i. e., independent of the choice of the utility representations; we abuse language by speaking of "the utility function v in $\mathcal{H}(A)$".

The graph of a concave and continuous function v on $\mathbb{R}_+^A$ is the envelope of its supporting hyperplanes, therefore it takes the form $v(y) = \min_{k\in K}\{\alpha_k \cdot y + \beta_k\}$ for some $\alpha_k \in \mathbb{R}^A, \beta_k \in \mathbb{R}$ and a possibly infinite set K. It is easy to see that v is also 1-homogeneous if and only if we can choose $\beta_k = 0$ for all k. So the simplest utilities in $\mathcal{H}(A)$ are additive, $v(y) = \alpha \cdot y$, and piecewise linear. For instance $A = \{a, b\}$ and $v(y) = \min\{y_a + y_b, 4y_a - y_b, 4y_b - y_a\}$, of which the indifference contours are represented on Figure 1. Note that this utility is not globally satiated, but for fixed y_b it is satiated at $y_a = y_b$. A smooth non linear and non monotonic function in $\mathcal{H}(A)$ is for example $v(y) = y_b \ln\{\frac{y_a}{y_b} + \frac{1}{2}\}$, represented in Figure 2.





Figures 1 and 2: examples of indifference contours.

A *fair division problem* is $\mathcal{P} = (N, A, u, \omega)$ where $u \in \mathcal{H}(A)^N$ is the profile of utility functions, and $\omega \in \mathbb{R}_+^A$ is the manna; we assume $\omega_a > 0$ for all a.

A *feasible allocation* (or simply an *allocation*) is $z \in \mathbb{R}_+^{N\times A}$ such that $\sum_N z_{ia} = \omega_a$ for all a, or in a more compact notation $z_N = \omega$. The corresponding utility profile is $U \in \mathbb{R}^N$ where $U_i = u_i(z_i)$. Let $\mathcal{F}(N, A, \omega)$ be the set of feasible allocations, and $\mathcal{U}(\mathcal{P})$ be the corresponding set of utility profiles. We always omit $\mathcal{P}$ or N, A if this creates no confusion.

We call a feasible utility profile U *efficient* if it is not Pareto dominated[4]; a feasible allocation is efficient if it implements an efficient utility profile.

Definition 1: *Given problem $\mathcal{P}$ a competitive division is a triple $(z \in \mathcal{F}, p \in \mathbb{R}^A, \beta \in \{-1, 0, +1\})$ where z is the competitive allocation, p is the competitive price and β the individual budget. The allocation z is feasible and each z_i maximizes i's utility in the budget set $B(p, \beta) = \{y_i \in \mathbb{R}_+^A | p\cdot y_i \leq \beta\}$:*

$$z_i \in d_i(p, \beta) = \arg \max_{y_i\in B(p,\beta)} \{u_i(y_i)\} \tag{1}$$

Moreover z_i minimizes i's wealth in her demand set

$$z_i \in \arg \min_{y_i\in d_i(p,\beta)} \{p \cdot y_i\} \tag{2}$$

[4]That is $U \leq U'$ and $U' \in \mathcal{U}(\mathcal{P}) \Longrightarrow U' = U$.

We write $CE(\mathcal{P})$ for the set of competitive allocations, and $CU(\mathcal{P})$ for the corresponding set of utility profiles.

Existence of a competitive allocation can be derived from (much) earlier results that do not require monotonic preferences (e.g., Mas-Colell (1982, Theorem 1) [23]; see also Shafer, Sonnenschein (1975) [35]); our main result in the next section gives instead a constructive proof.

In the normative perspective we adopt in this paper, we are only interested in efficient allocations. This is why to the usual demand property (1) we add (2) requiring demands to be *parsimonious*: each agent spends as little as possible for her competitive allocation. As already noted in Mas-Colell (1982) [23], if we omit (2) some satiated agents in N_- may inefficiently eat some items useless to themselves but useful to others.[5]

Recall three standard normative properties of an allocation $z \in \mathcal{F}(N, A, \omega)$. It is *Non Envious* iff $u_i(z_i) \geq u_i(z_j)$ for all i, j. It *Guarantees Fair Share* utility *iff* $u_i(z_i) \geq u_i(\frac{1}{n}\omega)$ for all i. It is in the *Weak Core from Equal Split* iff for all $S \subseteq N$ and all $y \in \mathbb{R}_+^{S \times A}$ such that $y_S = \frac{|S|}{n}\omega$, there is at least one $i \in S$ such that $u_i(z_i) \geq u_i(y_i)$. When we divide goods the competitive allocations meet these three properties, even in the much larger Arrow Debreu preference domain. This is still true when we divide mixed items.

Lemma 1 *A competitive allocation is efficient; it is Non Envious, Guarantees Fair Share, and is in the Weak Core from Equal Split.*

Proof. No Envy is clear. Fair Share Guaranteed holds because $B(p, \beta)$ contains $\frac{1}{n}\omega$. We give for completeness the standard argument for Efficiency. If (z, p, β) is a competitive division and z is Pareto-dominated by some $z' \in \mathcal{F}$, then for all $i \in N$ we must have $(p, z'_i) \geq (p, z_i)$ because otherwise i can either benefit or save money by switching to z'_i (property (2)). Since z' dominates z, some agent j strictly prefers z'_j to z_j, and therefore z'_j is outside his budget set, i.e., $(p, z'_j) > (p, z_j)$. Summing up these inequalities over all agents we get the contradiction $(p, \omega) > (p, \omega)$. The argument for the Weak Core property is similar. ■

Remark 1: A competitive allocation may fail the standard Core from Equal Split property, where coalition S blocks allocation z if it can use its endowment $\frac{|S|}{n}e^A$ to make everyone in S weakly better off and at least one agent strictly better. This is because "equal split" may give resources to agents who have no use for them.

4 Main result

We define formally the partition of division problems alluded to in the Introduction. Given a problem $\mathcal{P}$ we partition N as follows:

$$N_+ = \{i \in N | \exists z \in \mathcal{F} : u_i(z_i) > 0\} \ ; \ N_- = \{i \in N | \forall z \in \mathcal{F} : u_i(z_i) \leq 0\}$$

We call agents in N_+ *attracted to* the manna, and those in N_- *repulsed by* it. All agents in N_-, and only those, are globally satiated, and for them $z_i = 0$ is a global maximum, not necessarily unique.

[5] For instance $N = \{1, 2\}, A = \{a, b\}, \omega = (1, 1)$ and $u_1(z_1) = 6z_{1a} + 2z_{1b}$, $u_2(z_2) = -z_{2b}$. The inefficient allocation $z_1 = (\frac{1}{3}, 1), z_2 = (\frac{2}{3}, 0)$ meets (1) for $p = (\frac{3}{2}, \frac{1}{2})$ and $\beta = 1$. But $z'_2 = (0, 0)$ also gives zero utility to agent 2 and costs zero, so z_2 fails (2). The unique competitive division according Definition 1 is efficient: $z_1 = (1, 1), z_2 = (0, 0)$, and $p = (\frac{1}{2}, \frac{1}{2})$.

The partition is determined by the relative position of the set $\mathcal{U}$ of feasible utility profiles and the cone $\Gamma = \mathbb{R}_+^{N_+} \times \{0\}^{N_-}$, where attracted agents benefit while repulsed agents do not suffer. Write $\Gamma^* = \mathbb{R}_{++}^{N_+} \times \{0\}^{N_-}$for the relative interior of Γ.

Lemma 2 *Each problem $\mathcal{P}$ is of (exactly) one of three types:*
positive if $\mathcal{U} \cap \Gamma^* \neq \varnothing$; negative if $\mathcal{U} \cap \Gamma = \varnothing$; null if $\mathcal{U} \cap \Gamma = \{0\}$.

Note that the problem is positive if $u_i(\omega) > 0$ for at least one agent i. If we give nothing to a repulsed agent, an arbitrarily small allocation such that $u_j(z_j) > 0$ to each attracted agent j other than i, and the rest to i, the resulting utility profile is in Γ^* when z_j is small enough. The converse statement is not true: we give an example in the next section of a positive problem such that $u_i(\omega) < 0$ for all i.

Given a smooth function f and a closed convex set C we say that $x \in C$ *is a critical point of f in C* if the upper contour of f at x has a supporting hyperplane that supports C as well:

$$\forall y \in C : \partial f(x) \cdot y \leq \partial f(x) \cdot x \text{ and/or } \forall y \in C : \partial f(x) \cdot y \geq \partial f(x) \cdot x \tag{3}$$

This holds in particular if x is a local maximum or local minimum of f in C.

In the next statement we write $\mathcal{U}^{eff}$ for the set of efficient utility profiles, and $\mathbb{R}_=^N$ for the interior of $\mathbb{R}_-^N$.

Theorem *Competitive divisions exist in all problems $\mathcal{P}$. Moreover*
i) If $\mathcal{P}$ is positive their budget is $+1$; an allocation is competitive iff its utility profile maximizes the product $\Pi_{N_+} U_i$ over $\mathcal{U} \cap \Gamma^*$; so $CU(\mathcal{P})$ contains a single utility profile, positive in N_+ and null in N_-.
ii) If $\mathcal{P}$ is negative their budget is -1; an allocation is competitive iff its utility profile is in $\mathcal{U}^{eff} \cap R_=^N$ and is a critical point of the product $\Pi_N |U_i|$ in $\mathcal{U}$; so all utility profiles in $CU(\mathcal{P})$ are negative.
iii) $\mathcal{P}$ has a competitive division with a zero budget iff it is null; an allocation is competitive iff its utility profile is 0.

We see that the competitive utility profiles are entirely determined by the set of feasible utility profiles: the competitive approach still has a welfarist interpretation when we divide a mixed manna.

Moreover the Theorem implies that the task of dividing the manna is either good news (at least weakly) for everyone, or strictly bad news for everyone.

Remark 2: *The Competitive Equilibrium with Fixed Income Shares (CEFI for short) replaces in Definition 1 the common budget β by individual budgets $\theta_i \beta$, where the positive weights θ_i are independent of preferences. It is well known that in an all goods problem, this asymmetric generalization of the competitive solution obtains by maximizing the weighted product $\Pi_N U_i^{\theta_i}$ of utilities, so that it preserves the uniqueness, computational and continuity properties of the symmetric solution. The same is true of our Theorem, that remains valid word for word for the CEFI divisions upon raising $|U_i|$ to the power θ_i. In particular the partition of problems in positive, negative or null is unchanged.*

5 Examples and the multiplicity issue

We restrict attention in this section to the simple subdomain $\mathcal{L}(A)$ of $\mathcal{H}(A)$ where utilities are additive and represent linear preferences. Recall that the online platforms discussed in the Introduction work only in $\mathcal{L}(A)$, and that the companion paper provides additional normative results about competitive division in this domain.

5.1 Additive utilities

An additive utility function is a vector $u_i \in \mathbb{R}^A$. We write $U_i = u_i \cdot z_i = \sum_A u_{ia} z_{ia}$ for the corresponding utility at allocation z_i. For agent i item a is a good (resp. a bad) if $u_{ia} > 0$ (resp. $u_{ia} < 0$); if $u_{ia} = 0$ she is satiated with any amount of a. Given a problem $\mathcal{P}$ we partition A as follows

$$A_+ = \{a | \exists i : u_{ia} > 0\} \; ; \; A_- = \{a | \forall i : u_{ia} < 0\} \; ; \; A_0 = \{a | \max_i u_{ia} = 0\} \tag{4}$$

We call an item in A_+ a *collective good*, one in A_- a *collective bad*, and one in A_0 a *neutral* item. In an efficient allocation an item in A_+ is consumed only by agents for whom it is a good, and one in A_0 is consumed only by agents who are indifferent to it. This partition determines the sign of competitive prices.

If (z, p, β) *is a competitive division, we have*

$$p_a > 0 \text{ if } a \in A_+ \; ; \; p_a < 0 \text{ if } a \in A_- \; ; \; p_a = 0 \text{ if } a \in A_0 \tag{5}$$

The proof is simple. If the first statement fails an agent who likes a would demand an infinite amount of it; if the second one fails nobody would demand b. If the third one fails with $p_a > 0$ the only agents who demand a have $u_{ia} = 0$, which violates parsimony (2); if it fails with $p_a < 0$ an agent such that $u_{ia} = 0$ gets an arbitrarily cheap demand by asking large amounts of a, so (2) fails again. ■

Suppose a collective good $a \in A_+$ is a bad for agent i, $u_{ia} < 0$, and let z be a competitive allocation. Then i consumes no a, $z_{ia} = 0$, moreover if we only replace u_{ia} by $u'_{ia} = 0$, ceteris paribus, then z is still competitive in the new problem. This is why it is without loss of generality that in the examples below a collective good is never a bad for anyone: $a \in A_+ \implies u_{ia} \geq 0$ for all i.

Another innocuous assumption is that the manna has one unit of each item.

5.2 Two items

5.2.1 One good, one bad

The simplest mixed manna problem is easy to solve. Assume $A = \{a, b\}$ and $u_{ib} < 0 < u_{ia}$ for all i, then label the agents so that $\frac{u_{ia}}{|u_{ib}|}$ decreases with i: agent 1 is the least averse to b relative to a. The problem is positive *iff* $\frac{u_{1a}}{|u_{1b}|} > 1$. In this case the competitive division gives all of b and some of a to agent 1, and the same smaller share of a to agents $2, \cdots, n$.

The problem is null *iff* $\frac{u_{1a}}{|u_{1b}|} = 1$ (agent 1 eats the entire manna) and negative *iff* $\frac{u_{1a}}{|u_{1b}|} < 1$. In the latter case the competitive division is still unique (utilitywise) and has agent 1 eating all a and some b while others eat the same smaller share of b. Here is an example with two bads and three agents:

$$\mathcal{P}: \begin{array}{ccc} & a & b \\ u_1 & 3 & -4 \\ u_2 & 2 & -3 \\ u_3 & 1 & -2 \end{array} \quad \rightarrow \quad CE(\mathcal{P}): \begin{array}{ccc} & a & b \\ z_1 & 1 & \frac{5}{6} \\ z_2 & 0 & \frac{1}{12} \\ z_3 & 0 & \frac{1}{12} \end{array}$$

Note that in all three cases agent 1 ends up exactly at her Fair Share utility level $u_i \cdot \frac{1}{n}\omega$, while the other agents get strictly more.

5.2.2 Two bads

Computing competitive allocations is easy in this case because they are aligned in the one-dimensional set of efficient and envy-free allocations.

We assume for simplicity that all ratios $\frac{u_{ia}}{u_{ib}}$ are different, and label the agents $\{1, \cdots, n\}$ so that $\frac{u_{ia}}{u_{ib}}$ increases with i. Efficiency means that if i consumes some a and j consumes some b, then $i \leq j$; therefore at most one agent consumes both bads. Let z^i be the allocation where the agents in $\{1, \cdots, i\}$ share a (each gets $\frac{1}{i}$ unit) while those in $\{i+1, \cdots, n\}$ share b. It is efficient, and all envy-free allocations sit on the line $\cup_{i=1}^{n-1}[z^i, z^{1+1}]$. It is easy to check that z^i is envy-free *iff* i and $i+1$ do not envy each other, and in that case it is a competitive allocation as well. There may be other such allocations in the open intervals $]z^j, z^{j+1}[$. The tedious but straightforward computations are detailed in Proposition 4 of the companion paper: they imply that the upper bound on the cardinality of $CU(\mathcal{P})$ is $2n-1$.

Here is an example with $n = 2$ where $|CU(\mathcal{P})| = 3$:

$$\begin{array}{lccc} & & a & b \\ \mathcal{P}: & u_1 & -3 & -1 \\ & u_2 & -1 & -2 \end{array} \quad (6)$$

$$CE(\mathcal{P}): \begin{array}{ccc} & a & b \\ z_1 & \frac{1}{3} & 1 \\ z_2 & \frac{2}{3} & 0 \end{array}, \quad \begin{array}{ccc} & a & b \\ z_1' & 0 & 1 \\ z_2' & 1 & 0 \end{array}, \quad \begin{array}{ccc} & a & b \\ z_1'' & 0 & \frac{3}{4} \\ z_2'' & 1 & \frac{1}{4} \end{array}$$

and the corresponding utility profiles are depicted on Figure 3. Note that agent 1 (resp. agent 2) only gets his Fair Share utility at z (resp. at z'').

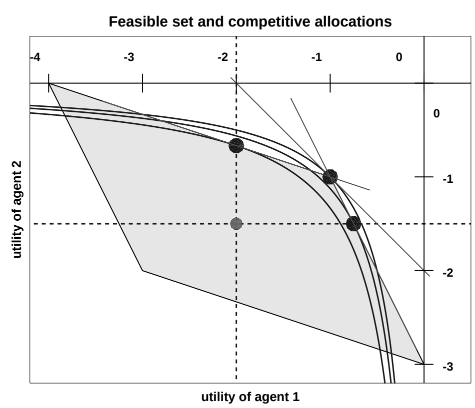


Figure 3: black dots represent the competitive utility profiles, the gray dot corresponds to the equal split of the manna in which both agents get their Fair Share utilities.

5.3 Two agents, goods and bads

Two agent problems are prominent in practice: they make for more than half of the visits on the SPLIDDIT website, and ADJUSTED WINNER is designed only for those problems. We explain after the examples why the competitive allocations are still easy to compute.

Let us add a good c to the problem $\mathcal{P}$ from subsection 5.2.2. Assuming that c is worth $\lambda > 0$ for both agents we get the following family of problems:

$$\begin{array}{rcccc} & & a & c & b \\ \mathcal{P}(\lambda): & u_1 & -3 & \lambda & -1 \\ & u_2 & -1 & \lambda & -2 \end{array}.$$

When λ goes from 4 to 1, the problem from positive becomes null ($\lambda = 2$) and then negative thereby illustrating all the patterns described by the Theorem. Figure 4 shows the competitive utility profiles for three representative values of λ: 4, 2 and 1. And Figure 3 is for $\lambda = 0$.

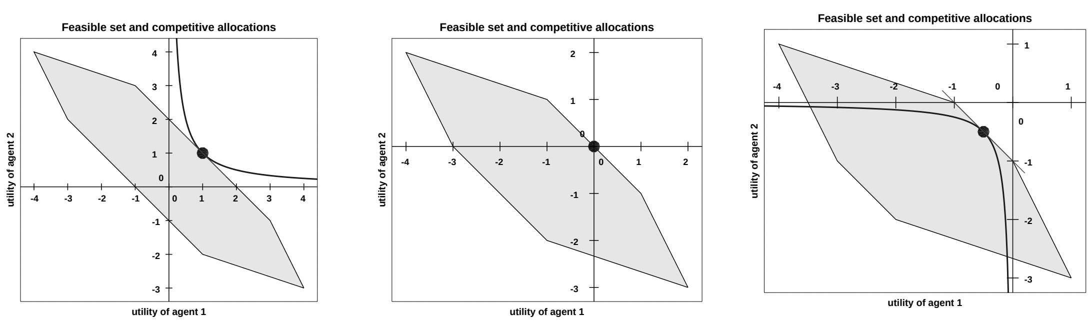


Figure 4: positive ($\lambda = 4$), null ($\lambda = 2$), and negative problems ($\lambda = 1$).

As noted after Lemma 2, the problem is positive if $u_i \cdot \omega > 0$ for one of $i = 1, 2$: this is true here for $\lambda > 3$. But the problem is still positive for $2 < \lambda \leq 3$, despite the fact that $u_i \cdot \omega \leq 0$ for both agents.

In a two agent problem the set of efficient utility profiles is the one-dimensional frontier of a two-dimensional polytope. Recall that each item a is either a good for both agents or a bad for both, and assume for simplicity that all ratios $\frac{u_{1a}}{u_{2a}}$ are different. We label the items as $A = \{1, \cdots, m\}$ ensuring that $\frac{u_{1k}}{u_{2k}}$ decreases with k. For $k \in \{0, 1, \cdots, m\}$ let z^k be the allocation where agent 1 gets all goods in $\{1, \cdots, k\}$ (or no good if $k = 0$), and all bads in $\{k+1, \cdots, m\}$ (or no bad if $k = m$); agent 2 eats the rest, i. e., all goods in $\{k+1, \cdots, m\}$ and all bads in $\{1, \cdots, k\}$. It is easy to check that the broken line $\cup_{k=0}^{m}[z^k, z^{k+1}]$ is precisely the set of efficient allocations, and fairly straightforward to compute all competitive allocations from the sequence of ratios $\frac{u_{1k}}{u_{2k}}$. It follows that the upper bound on $|CU(\mathcal{P})|$ is $2m - 1$ (statement *iii*) in Theorem 1 of the companion paper).

5.4 Counting competitive allocations

In problems with three or more agents and items, it is no longer elementary to compute the set of competitive allocations. We know that the number $|CU(\mathcal{P})|$ of different competitive utility profiles is always finite, and that its upper bound grows exponentially in n, m: $|CU(\mathcal{P})|$ *can be as high as*

$2^{\min\{n,m\}} - 1$ *if* $n \neq m$, *and* $2^{n-1} - 1$ *if* $n = m$ (statement *iii*) in Theorem 1 of the companion paper). A 6×5 example illustrating this fact is given in Subsection 8.3.

We also check in the companion paper that if $n = 2$ and/or $m = 2$, $|CU(\mathcal{P})|$ is odd in almost all problems (excluding only those where the coefficients of u satisfy certain simple equations). We conjecture that a similar statement holds for any n, m.

6 Single-valued competitive division rules

Without backing up this proposal by specific normative arguments, we submit that a natural selection of $CU(\mathcal{P})$ obtains by *maximizing* the product $\prod_{i \in N} |U_i|$ of individual *dis*utilities on the negative efficiency frontier.[6]

Lemma 3 *If* $\mathcal{P}$ *is a negative problem, the profile* U^* *maximizing the Nash product* $\prod_{i \in N} |U_i|$ *over* $\mathcal{U}^{eff} \cap \mathbb{R}^N_-$ *is a critical point of this product on* $\mathcal{U}$. *It is a competitive utility profile:* $U^* \in CU(\mathcal{P})$.

We prove Lemma 3 in Subsection 8.4 in the domain $\mathcal{H}(A)$. In the additive domain $\mathcal{L}(A)$ this selection is almost always unique.

Lemma 4 *Fix* N, A *and* ω. *For almost all negative problems* $\mathcal{P} = (N, A, u, \omega)$ *with additive utilities (w.r.t. the Lebesgue measure on the space* $R^{N \times A}$ *of utility matrices) the utility profile* U^* *defined in Lemma 3 is unique.*

In the 6×5 example of Subsection 8.3, U^* is the profile treating equally the first five agents. However in example (6) the three competitive utility profiles are $(-2, -\frac{2}{3})$, $(-1, -1)$, $(-\frac{3}{4}, -\frac{3}{2})$, and U^* is the first one where agent 1 gets his Fair Share -2. This is the typical (in fact generic) situation for two agent, two-bad problems where $|CU(\mathcal{P})| = 3$: our selection always picks one of the two extreme divisions, instead of the natural compromise where both agents improve upon their Fair Share.

Finally when $n = 2$ and/or $m = 2$, we mentioned in the previous Subsection that $|CU(\mathcal{P})|$ is generically odd. As the corresponding profiles are arranged along a broken line, a more natural selection (again, not supported by any normative argument) picks the median profile.

Whatever the selection we end up proposing, it will exhibit the two unpalatable features to which we now turn.

7 Two impossibility results

7.1 The continuity issue

We show that there is no single-valued selection $\mathcal{P} \rightarrow CU(\mathcal{P})$ continuous in the utility parameters. In fact we show a stronger discontinuity result about the much bigger correspondence of utility profiles at efficient and envy-free allocations.

Because Proposition 1, and Proposition 2 in the next subsection, deal with axioms comparing the allocations selected at different problems, we define first division rules. For the sake of brevity, we only give the definition in the additive domain $\mathcal{L}(A)$. Its extension to $\mathcal{H}(A)$ is straightforward.

Notation: when we rescale each utility u_i as $\lambda_i u_i$, the new utility matrix is written $\lambda * u$.

[6] Note that *minimizing* $\prod_{i \in N} |U_i|$ on $\mathcal{U} \cap \mathbb{R}^N_-$ picks all the boundary points where this product is null, neither a competitive allocation nor a meaningful division rule.

Definition 2 *A division rule f associates to every problem $\mathcal{P} = (N, A, u, \omega)$ a set of feasible allocations $f(\mathcal{P}) \subset \mathcal{F}(N, A, \omega)$ such that for any rescaling $\lambda \in \mathbb{R}^N_{++}$, we have: $f(N, A, \lambda * u, \omega) = f(N, A, u, \omega)$. Moreover f meets Pareto-Indifference (PI): for every $\mathcal{P}$ and $z, z' \in \mathcal{F}(N, A, \omega)$ we have*

$$\{z \in f(\mathcal{P}) \text{ and } u_i \cdot z_i = u_i \cdot z'_i \text{ for all } i\} \Longrightarrow z' \in f(\mathcal{P})$$

Property PI implies that f is entirely determined by its utility correspondence $\mathcal{P} \to F(\mathcal{P}) = \{u \cdot z | z \in f(\mathcal{P})\}$. Invariance to rescaling makes sure that division rules are ordinal constructs, they only depend upon the underlying linear preferences. The competitive division rule $\mathcal{P} \to CE(\mathcal{P})$ meets Definition 2.

Abusing language, we call the division rule f *single-valued* if the corresponding mapping F is single-valued.

We call the division rule f **Continuous** (CONT) if for each choice of N, A, and ω the corresponding mapping $u \to F(N, A, u, \omega)$ is upper-hemi-continuous in $\mathbb{R}^{N \times A}$. If the division rule does not depend upon the units of items in A,[7] CONT implies that $\mathcal{P} \to F(\mathcal{P})$ is also (upper-hemi) continuous in $\omega \in \mathbb{R}^A_+$.

We call the rule f **Envy-Free** (EVFR) if $f(\mathcal{P})$ contains at least one envy-free allocation for every problem $\mathcal{P}$.

Proposition 1 *With two or more bads and four or more agents, no single-valued rule can be Efficient, Envy-Free and Continuous.*

In particular the competitive rule CE, that is a continuous and envy-free correspondence, admits no continuous single-valued selection.

Proposition 1 is Theorem 2 in the companion paper, where it follows from the fact that when the manna contains two bads, the set of envy-free and efficient allocations can have $\lfloor \frac{2n+1}{3} \rfloor$ connected components (Proposition 4). In Section 8.6 we give for completeness a four agents, two bads example illustrating the key arguments.

This incompatibility result is tight. The equal division rule, $F_i(\mathcal{P}) = \{\frac{1}{n} u_i \cdot \omega\}$ for all $\mathcal{P}$, meets EVFR and CONT. A single-valued selection of the competitive rule CU meets EFF and EVFR. We leave the reader construct a rule meeting EFF and CONT.

7.2 Resource Monotonicity

Our last axiomatic property is often viewed as a compelling normative consequence of the common property of the resources we are dividing. Increasing in the manna the amount of a unanimous good (an item that everyone likes), or decreasing that of a unanimous bad, should not hurt anyone: welfare should be co-monotonic to the quality of the common resources. Early references are Roemer (1986) [31], Moulin, Thomson (1988) [26]; Thomson (2010) [40] is a recent survey.

We say that problem $\mathcal{P}'$ improves problem $\mathcal{P}$ on item $a \in A$ if $\mathcal{P}$ and $\mathcal{P}'$ only differ in $\omega_a \neq \omega'_a$ and either $\{\omega_a < \omega'_a$ and $u_{ia} \geq 0$ for all $i\}$ or $\{\omega_a > \omega'_a$ and $u_{ia} \leq 0$ for all $i\}$.

Resource Monotonicity (RM): if $\mathcal{P}'$ improves upon $\mathcal{P}$ on item $a \in A$, then $F(\mathcal{P}) \leq F(\mathcal{P}')$

Proposition 2

i) If we divide bads between two or more agents, no single-valued rule can be Efficient, Resource Monotonic and Guarantee Fair Share.

ii) The competitive rule to divide ***goods*** *is Resource Monotonic in the additive utility domain.*

[7] That is, for each $\lambda > 0$ the set $F(\mathcal{P})$ is unchanged if we replace ω_a by $\lambda \omega_a$ and each u_{ia} by $\frac{1}{\lambda} u_{ia}$. Clearly CU meets this property.

The proof of statement $i)$ is by means of a simple two-person, two-bad example. Fix a rule F meeting EFF, RM and GFS and consider the problem

$$\mathcal{P} : \omega = (1,1) \text{ and } \begin{array}{ccc} & a & b \\ u_1 & -1 & -4 \\ u_2 & -4 & -1 \end{array}$$

Set $U = F(\mathcal{P})$. As $(-1,-1)$ is an efficient utility profile, one of U_1, U_2 is at least -1, say $U_1 \geq -1$. Now let $\omega' = (\frac{1}{9}, 1)$ and pick $z' \in f(\mathcal{P}')$. By GFS and feasibility:

$$-z'_{2b} \geq u_2 \cdot z'_2 \geq \frac{1}{2} u_2 \cdot \omega' = -\frac{13}{18}$$

$$\Longrightarrow z'_{1b} \geq \frac{5}{18} \Longrightarrow u_1 \cdot z'_1 = U'_1 \leq -\frac{10}{9} < U_1$$

contradicting RM. Extending this argument to the general case $n \geq 2, m \geq 2$ is straightforward.

We omit for brevity the proof of statement $ii)$, available as Proposition 2in the companion paper and in Segal-Halevi, Sziklai (2015) [33] for the more general cake-division model. This statement generalizes easily to positive problems, when we add a unanimous good to an already positive problem.

We stress that this positive result applies only to the domain $\mathcal{L}(A)$, it does not extend to general monotone 1-homogenous utilities in $\mathcal{H}(A)$. On the latter domain, precisely the same combination of axioms as in Proposition 2 cannot be together satisfied: see Moulin, Thomson (1988) [26] and Thomson, Kayi (2005) [41]. This makes the contrast between goods and bads problems in $\mathcal{L}(A)$ all the more stark.

8 Proofs

8.1 Lemma 2

The three cases are clearly mutually exclusive; we check they are exhaustive. It is enough to show that if $\mathcal{U}$ intersects $\Gamma_{\neq 0} = \Gamma \diagdown \{0\}$ then it intersects Γ^* as well. Let $z \in \mathcal{F}$ be an allocation with $u(z) \in \Gamma_{\neq 0}$ and i_+ be an agent with $u_{i_+}(z_{i_+}) > 0$. Define a new allocation z' with $z'_{i_+} = z_{i_+} + \varepsilon \sum_{j \neq i_+} z_j$ and $z'_j = (1-\varepsilon) z_j$ for $j \neq i_+$. By continuity we can select a small $\varepsilon > 0$ such that $u(z') \in \Gamma_{\neq 0}$. By construction $z'_{i_+ a} > 0$ for all $a \in A$.

For any $j \in N_+ \setminus \{i_+\}$ we can find $y_j \in \mathbb{R}^A$ such that $u_j(z'_j + \delta y_j) > 0$ for small $\delta > 0$. Indeed if $u_j(z'_j)$ is positive we can take $y_j = 0$. And if $u_j(z'_j) = 0$, assuming that y_j does not exist implies that z'_j is a local maximum of u_j. By concavity of u_j it is then a global maximum as well, which contradicts the definition of N_+.

Consider an allocation z'': $z''_{i_+} = z'_{i_+} - \delta \sum_{j \in N_+ \setminus \{i_+\}} y_j$, $z''_j = z'_j + \delta y_j$ for $j \in N_+ \setminus \{i_+\}$ and $z''_k = z'_k$ for $k \in N_-$. For small $\delta > 0$ this allocation is feasible and yields utilities in Γ^*.

8.2 Main Theorem

Throughout the proof it is convenient to consider competitive divisions (z, p, β) with arbitrary budgets $\beta \in \mathbb{R}$ (not only $\beta \in \{-1, 0, 1\}$); this clearly yields exactly the same set of competitive allocations $CE(\mathcal{P})$ and utility profiles $CU(\mathcal{P})$.

8.2.1 Positive problems: statement $i)$

Let $\mathcal{N}(V) = \prod_{i \in N_+} V_i$ be the Nash product of utilities of the attracted agents. We fix a positive problem $\mathcal{P}$ and proceed in two steps.

Step 1. *If U maximizes $\mathcal{N}(V)$ over $V \in \mathcal{U} \cap \Gamma^*$ and $z \in \mathcal{F}$ is such that $U = u(z)$, then z is a competitive allocation with budget $\beta > 0$.*
Let $\mathcal{C}_+$ be the convex cone of all $y \in \mathbb{R}_+^{N \times A}$ with $u(y) \in \Gamma$. For any $\lambda > 0$ put

$$\mathcal{C}_\lambda = \left\{ y \in \mathcal{C}_+ \mid \mathcal{N}(u(y)) \geq \lambda^{|N_+|} \right\}.$$

Since $\mathcal{P}$ is positive the set $\mathcal{C}_\lambda$ is non-empty for any $\lambda > 0$. Continuity and concavity of utilities imply that $\mathcal{C}_\lambda$ is closed and convex. Homogeneity of utilities give $\mathcal{C}_\lambda = \lambda \mathcal{C}_1$.

Set $\lambda^* = (\mathcal{N}(U))^{\frac{1}{|N_+|}}$. The set $\mathcal{C}_\lambda$ does not intersect $\mathcal{F}$ for $\lambda > \lambda^*$, and $\mathcal{C}_{\lambda^*}$ touches $\mathcal{F}$ at z.
Step 1.1 There exists a hyperplane H separating $\mathcal{F}$ from $\mathcal{C}_{\lambda^*}$.
Consider a sequence λ_n converging to λ^* from above. Since $\mathcal{C}_{\lambda_n}$ and $\mathcal{F}$ are convex sets that do not intersect, they can be separated by a hyperplane H_n. The family $\{H_n\}_{n \in \mathbb{N}}$ has a limit point H. The hyperplane H separates $\mathcal{F}$ from $\mathcal{C}_{\lambda^*}$ by continuity of u. Thus there exist $q \in \mathbb{R}^{N \times A}$ and $Q \in \mathbb{R}$ such that $\sum_{i,a} q_{ia} y_{ia} \leq Q$ for $y \in \mathcal{F}$ and $\sum_{i,a} q_{ia} y_{ia} \geq Q$ on $\mathcal{C}_{\lambda^*}$. The coefficients q_{ia} will be used to define the vector of prices p.

By the construction z maximizes $\mathcal{N}(u(y))$ over $\mathcal{B}^N(q, Q) = \{y \in \mathcal{C}_+ \mid \sum_{i,a} q_{ia} y_{ia} \leq Q\}$. Think of the latter as a "budget set with agent-specific prices".

Define the vector of prices p by $p_a = \max_{i \in N} q_{ia}$ and $\mathcal{B}^*(p, Q) = \{y \in \mathcal{C}_+ \mid \sum_i p \cdot y_i \leq Q\}$. We show now that we do not need agent-specific pricing.
Step 1.2 The allocation z maximizes $\mathcal{N}(u(y))$ over $y \in \mathcal{B}^*(p, Q)$.
It is enough to show the double inclusion $z \in \mathcal{B}^*(p, Q) \subset \mathcal{B}^N(q, Q)$. The second one is obvious since $\sum_{i,a} y_{ia} p_a \leq Q$ implies $\sum_{i,a} y_{ia} q_{ia} \leq Q$. Let us check the first inclusion. Taking into account that $z \in \mathcal{F}$ and $\sum_{i,a} q_{ia} y_{ia} \leq Q$ for $y \in \mathcal{F}$, we get

$$\sum_i p \cdot z_i = \sum_a p_a \sum_i z_{ia} = \sum_a p_a = \sum_a \max_i q_{ia} = \max_{y \in \mathcal{F}} \sum_{i,a} q_{ia} y_{ia} \leq Q.$$

Step 1.3 (z, p, β) is a competitive division for some $\beta > 0$.
Consider an agent i from N_+. Check that the bundle z_i belongs to his competitive demand $d_i(p, \beta_i)$, where $\beta_i = p \cdot z_i$. Indeed if there exists $z_i' \in \mathbb{R}_+^A$ such that $p \cdot z_i' \leq \beta_i$ and $u_i(z_i') > u_i(z_i)$, then switching the consumption of agent i from z_i to z_i' gives an allocation in $\mathcal{B}^*(p, Q)$ and increases the Nash product, contradicting Step 1.2. Note that $\beta_i > 0$ for $i \in N_+$ because otherwise we can take $z_i' = 2 z_i$. Check now that z_i is parsimonious: it minimizes $p \cdot y_i$ over $d_i(p, \beta_i)$. If not, pick $y_i \in d_i(p, \beta_i)$ with $p \cdot y_i < p \cdot z_i$, then for δ small enough and positive, the bundle $z_i' = (1 + \delta) y_i$ meets $p \cdot z_i' \leq \beta_i$ and $u_i(z_i') > u_i(z_i)$.

We use now the classic equalization argument (Eisenberg (1961) [12]) to check that β_i does not depend on $i \in N_+$. We refer to the fact that the geometric mean is below the arithmetic one as "the inequality of means".

Assume $\beta_i \neq \beta_j$ and consider a new allocation z', where the budgets of i and j are equalized: $z_i' = \frac{\beta_i + \beta_j}{2\beta_i} z_i$ and $z_j' = \frac{\beta_i + \beta_j}{2\beta_j} z_j$. This allocation belongs to $\mathcal{B}^*(p, Q)$ and homogeneity of utilities implies

$$\mathcal{N}(u(z')) = \mathcal{N}(U) \left(\frac{\beta_i + \beta_j}{2\beta_i} \right) \left(\frac{\beta_i + \beta_j}{2\beta_j} \right).$$

Now the (strict) inequality of means gives $\frac{\beta_i+\beta_j}{2} > \sqrt{\beta_i\beta_j}$, therefore $\mathcal{N}(u(z')) > \mathcal{N}(U)$ contradicting the optimality of z. Denote the common value of β_i by β.

Turning finally to the repulsed agents we check that for any $i \in N_-$ there is no z'_i such that $u_i(z'_i) = 0$ and $\beta'_i = p \cdot z'_i < p \cdot z_i = \beta_i$, i.e., i can not decrease his spending. Assuming that z'_i exists we can construct an allocation $z' \in \mathcal{B}^*(p, Q)$, where agent i switches to z'_i, consumption of other agents from N_- remains the same, and $z'_j = z_j \frac{N_+\beta+\beta_i-\beta'_i}{N_+\beta}$ for $j \in N_+$. In other words, money saved by i are redistributed among positive agents. By homogeneity $\mathcal{N}(u(z')) > \mathcal{N}(U)$, contradiction. A corollary is that β_i must be zero: take $z'_i = 0$ if $\beta_i > 0$, and $z'_i = 2z_i$ if β_i is negative. At z_i agent i reaches his maximal welfare of zero. Therefore, if i can afford z_i, then z_i is in the demand set. Since the price β_i of z_i is zero, we conclude $z_i \in d_i(p, \beta)$. The proof of Step 1.3 and of Step 1 is complete.

Step 2 *If (z, p, β) is a competitive division, then $\beta > 0$, and $U = u(z)$ belongs to $\mathcal{U} \cap \Gamma^*$ and maximizes $\mathcal{N}$ over this set.*
Check first $\beta > 0$. If $\beta \leq 0$ the budget set $B(p, \beta)$ contains z_i and $2z_i$ for all i, therefore $u_i(2z_i) \leq u_i(z_i)$ implies $U_i \leq 0$. Then U is Pareto-dominated by any $U' \in \mathcal{U} \cap \Gamma^*$, contradicting the efficiency of z (Lemma 1).

Now $\beta > 0$ implies U belongs to Γ^*: every $i \in N_+$ has a y_i with $u_i(y_i) > 0$ and can afford δy_i for small enough $\delta > 0$; every $i \in N_-$ can afford $y_i = 0$, hence $u_i(z_i) = 0$ and $p \cdot z_i \leq 0$ (by (2)).

Consider $U' = u(z')$ that maximizes $\mathcal{N}$ over $\mathcal{U} \cap \Gamma^*$. For any $i \in N_+$ his spending $\beta'_i = p \cdot z'_i$ must be positive. Otherwise $\delta z'_i \in B(p, \beta)$ for any $\delta > 0$ and agent i can reach unlimited welfare. Similarly $\beta'_i < 0$ for $i \in N_-$ implies $\delta z'_i \in d_i(p, \beta)$ for any $\delta > 0$, so the spending in $d_i(p, \beta)$ is arbitrarily low, in contradiction of parsimony (2).

For attracted agents $\frac{\beta}{\beta'_i} z'_i \in B(p, \beta)$ gives $\frac{\beta}{\beta'_i} U'_i = u_i\left(\frac{\beta}{\beta'_i} z'_i\right) \leq U_i$. Therefore if U is not a maximizer of $\mathcal{N}$, we have

$$\mathcal{N}(U) < \mathcal{N}(U') \leq \mathcal{N}(U) \prod_{i \in N_+} \frac{\beta'_i}{\beta} \Longrightarrow 1 < \left(\prod_{i \in N_+} \frac{\beta'_i}{\beta}\right)^{\frac{1}{|N_+|}} \leq \frac{\sum_{i \in N_+} \beta'_i}{|N_+|\beta}$$

where we use again the inequality of means. Now we get a contradiction from

$$\sum_{i \in N_+} \beta'_i \leq \sum_{i \in N} \beta'_i = p \cdot \omega = \sum_{i \in N} p \cdot z_i \leq \sum_{i \in N_+} \beta + \sum_{i \in N_-} 0 = |N_+|\beta$$

8.2.2 Negative problems: statement *ii*)

The proof is simpler because we do not need to distinguish agents from N_+ and N_-. We define the Nash product for negative problems by $\mathcal{N}(V) = \prod_{i \in N} |V_i|$ and focus now on its critical points in $\mathcal{U}^{eff}$. We start by the variational characterization of such points. If $V \in \mathbb{R}^N_=$ we have $\frac{\partial}{\partial V_i} \mathcal{N}(V) = \frac{1}{V_i} \mathcal{N}(V)$. Therefore $U \in \mathcal{U} \cap \mathbb{R}^N_=$ is a critical point of $\mathcal{N}$ on $\mathcal{U}$ that lay on $\mathcal{U}^{eff}$ iff

$$\sum_{i \in N} \frac{U'_i}{|U_i|} \leq -|N| \quad \text{for all } U' \in \mathcal{U} \tag{7}$$

The choice of the sign in this inequality is determined by Efficiency. Set $\varphi_U(U') = \sum_{i \in N} \frac{U'_i}{|U_i|}$: inequality (7) says that $U' = U$ maximizes $\varphi_U(U')$ on $\mathcal{U}$.

We fix a negative problem $\mathcal{P}$ and proceed in two steps.

Step 1. *If a utility profile $U \in \mathcal{U}^{eff} \cap \mathbb{R}^N_=$ is a critical point of $\mathcal{N}$ on $\mathcal{U}$, then any $z \in \mathcal{F}$ implementing U is a competitive allocation with budget $\beta < 0$.*
By (7) for any $y \in \mathcal{F}$ we have $\varphi_U(u(y)) \leq -|N|$. Define

$$\mathcal{C}_\lambda = \left\{ y \in \mathbb{R}^{N\times A}_+ \mid \varphi_U(u(y)) \geq \lambda \right\}$$

For $\lambda \leq 0$ it is non-empty (it contains 0), closed and convex. For $\lambda > -|N|$ the set $\mathcal{C}_\lambda$ does not intersect $\mathcal{F}$ and for $\lambda = -|N|$ it touches $\mathcal{F}$ at z. Consider a hyperplane $\sum_{i,a} q_{ia} y_{ia} = Q$ separating $\mathcal{F}$ from $\mathcal{C}_{-|N|}$ and fix the sign by assuming $\sum_{i,a} q_{ia} y_{ia} \leq Q$ on $\mathcal{F}$ (existence follows as in Step 1.1 for positive problems). By the construction z maximizes $\varphi_U(u(y))$ on $\mathcal{B}^N(q,Q) = \{y \in \mathbb{R}^{N\times A}_+ \mid \sum_{i,a} q_{ia} y_{ia} \leq Q\}$. Defining prices by $p_a = \max_{i\in N} q_{ia}$ and mimicking the proof of Step 1.2 for positive problems we obtain that z belongs to $\mathcal{B}^*(p,Q) = \{y \in \mathbb{R}^{N\times A}_+ \mid \sum_{i,a} p_a y_{ia} \leq Q\}$ and maximizes $\varphi_U(u(y))$ there.

We check now that z is a competitive allocation with negative budget. For any agent $i \in N$ the bundle z_i belongs to his demand $d_i(p, \beta_i)$ (as before $\beta_i = p \cdot z_i$). If not, i can switch to any $z'_i \in B(p, \beta_i)$ with $u_i(z'_i) > U_i$, thus improving the value of φ_U and contradicting the optimality of z. The maximal spending β_i must be negative, otherwise i can afford $y_i = 0$ and $u_i(z_i) < u_i(y_i)$. If there is some $z'_i \in d_i(p,\beta_i)$ such that $p \cdot z'_i < \beta_i$, the bundle $z''_i = \frac{\beta_i}{p\cdot z'_i} z'_i$ is still in $B(p,\beta_i)$ and $u_i(z''_i) > U_i$: therefore $p \cdot z_i = \beta_i$ and z_i is parsimonious ((2)).

Finally, $\beta_i = \beta_j$ for all $i, j \in N$. If $\beta_i \neq \beta_j$, we use an *un*equalization argument dual to the one in Step 1.3 for positive problems. Assume for instance $\beta_i > \beta_j \Leftrightarrow |\beta_i| < |\beta_j|$ and define z' from z by changing only z'_i to $\frac{1}{2} z_i$ and z'_j to $\frac{2\beta_j + \beta_i}{2\beta_j} z_j$. Clearly $z' \in \mathcal{B}^*(p,Q)$ and we compute

$$\varphi_U(u(z')) - \varphi_U(U) = -\frac{1}{2} - \frac{2\beta_j + \beta_i}{2\beta_j} + 2 = \frac{1}{2} - \frac{\beta_i}{2\beta_j} > 0$$

But we showed that z maximizes $\varphi_U(u(y))$ in $\mathcal{B}^*(p,Q)$: contradiction.

Step 2. *If (z, p, β) is a competitive division, then $\beta < 0$ and the utility profile $U = u(z)$ is a critical point of the Nash product on U that belongs to $\mathcal{U}^{eff} \cap \mathbb{R}^N_=$.*
Check first that $\beta < 0$. If not each agent can afford $y_i = 0$ so $U_i \geq 0$ for all i, which is impossible in a negative problem. Assume next $U_i \geq 0$ for some i: we have $2z_i \in B(p,\beta)$, $u_i(2z_i) \geq u_i(z_i)$, and $p \cdot (2z_i) < p \cdot z_i$, which contradicts (1) and/or (2) in Definition 1. Therefore U belongs to $\mathbb{R}^N_=$. Finally $p \cdot z_i < \beta$ would imply $u_i(z_i) < u_i(\lambda z_i)$ for $\lambda \in [0, 1[$, and $\lambda z_i \in B(p,\beta)$ for λ close enough to 1, a contradiction. Summarizing we have shown $U \in \mathcal{U}^{eff} \cap \mathbb{R}^N_=$ and $p \cdot z_i = \beta < 0$ for all i.

To prove that U is a critical point it is enough to check that it maximizes $\varphi_U(u(y))$ on $\mathcal{F}$. Fix $z' \in \mathcal{F}$, set $U' = u(z')$ and $p \cdot z'_i = \beta'_i$. To show $\varphi_U(U') \leq \varphi_U(U)$ we will prove

$$U'_i \leq \frac{\beta'_i}{\beta} U_i \text{ for all } i \tag{8}$$

This holds if $\beta'_i < 0$ because $\frac{\beta}{\beta'_i} z'_i \in B(p,\beta)$ so $\frac{\beta}{\beta'_i} U'_i = u_i\left(\frac{\beta}{\beta'_i} z'_i\right) \leq U_i$. If $\beta'_i \geq 0$ we set $z''_i = \alpha z'_i + (1-\alpha) z_i$, where $\alpha > 0$ is small enough that $p \cdot z''_i < 0$. We just showed (8) holds for $u_i(z''_i)$, therefore

$$u_i(z''_i) \leq \frac{p \cdot z''_i}{\beta} U_i = \alpha \frac{\beta'_i}{\beta} U_i + (1-\alpha) U_i.$$

Concavity of u_i gives $\alpha U_i' + (1-\alpha)U_i \leq u_i(z_i'')$ and the proof of (8) is complete. Now we sum up these inequalities and reach the desired conclusion

$$\varphi_U(U') = \sum_{i\in N} \frac{U_i'}{|U_i|} \leq -\frac{\sum_{i\in N}\beta_i'}{\beta} = -\frac{\sum_{i\in N} p\cdot z_i'}{\beta} = -\frac{p\cdot\omega}{\beta} = -|N| = \varphi_U(U)$$

8.2.3 Null problems: statement *iii*)

The proof resembles that for positive problems, as we must distinguish N_+ from N_-, but the Nash product no longer plays a role. Fix a null problem $\mathcal{P}$.

Step 1. *Any $z \in \mathcal{F}$ such that $u(z) = 0$ is competitive with $\beta = 0$.*

Suppose first all agents are repulsed, $N = N_-$. Then $u_i(y_i) \leq 0$ for all $i \in N$ and $y_i \in \mathbb{R}_+^A$ and $(z, 0, 0)$ is a competitive division: everybody has zero money, all bundles are free and all agents achieve the best possible welfare with the smallest possible spending. We assume from now on $N_+ \neq \varnothing$.

Define $\psi(y) = \min_{i\in N_+} u_i(y_i)$ for $y \in \mathbb{R}_+^{N\times A}$ and the sets $\mathcal{C}_\lambda = \{y \in \mathcal{C}_+ \mid \psi(y) \geq \lambda\}$, where $\mathcal{C}_+ = \{y \in \mathbb{R}_+^{N\times A} | u(y) \in \Gamma\}$ (as in the positive proof). For $\lambda \geq 0$ the set $\mathcal{C}_\lambda$ is non-empty, closed and convex. If $\lambda > 0$, the sets $\mathcal{C}_\lambda$ and $\mathcal{F}$ do not intersect. As in Step 1.1 of the positive proof we construct a hyperplane separating $\mathcal{F}$ and $\mathcal{C}_0$, define the set $\mathcal{B}^N(q, Q)$, the vector of prices p, and the set $\mathcal{B}^*(p, Q)$. Similarly we check that the allocation z maximizes $\psi(y)$ over $y \in \mathcal{B}^*(p, Q)$, and $\psi(z) = 0$.

We set $\beta_i = p \cdot z_i$ and show that $(z, p, 0)$ is a competitive division in three substeps.

Step 1.1 for all $i \in N$ and $x_i \in \mathbb{R}_+^A$: $p \cdot x_i < \beta_i \Longrightarrow u_i(x_i) < 0$.

Suppose $p \cdot x_i < \beta_i$ and $u_i(x_i) \geq 0$ for some $i \in N_+$. For each j in N_+ pick a bundle y_j^+ such that $u_j(y_j^+) > 0$ and construct the allocation z' as follows: $z_i' = x_i + \delta y_i^+$; $z_j' = z_j + \delta y_j^+$ for any other $j \in N_+$; $z_j' = z_j$ for $j \in N_-$. If $\delta > 0$ is small enough $z' \in \mathcal{B}^*(p, Q)$ and for any $j \in N_+$ we have $u_j(z_j') > 0$, by concavity and homogeneity of u_j. For instance

$$\frac{1}{2}u_i(z_i') = u_i\left(\frac{1}{2}x_i + \frac{1}{2}\delta y_i^+\right) \geq \frac{1}{2}u_i(x_i) + \frac{1}{2}\delta u_i(y_i^+) > 0$$

Therefore $\psi(z') > 0$ contradicting the optimality of z.

The proof when $p \cdot x_i < \beta_i$ and $u_i(x_i) \geq 0$ for some $i \in N_-$ is similar and left to the reader.

Step 1.2 $\beta_i = 0$ for all $i \in N$.

If $\beta_i > 0$ then $x_i = 0$ is such that $p \cdot x_i < \beta_i$ and $u_i(x_i) = 0$, which we just ruled out. If $\beta_i < 0$ then $p \cdot (2z_i) < \beta_i$ yet $u_i(2z_i) = 0$, contradicting Step 1.1.

From Steps 1.1, 1.2 we see that for all i if $y_i \in d_i(p, 0)$ then $p \cdot y_i = 0$: so if we show $z_i \in d_i(p, 0)$ the parsimony property (2) is automatically satisfied. Therefore our next substep completes the proof of Step 1.

Step 1.3 $z_i \in d_i(p, 0)$ for all $i \in N$.

For $i \in N_-$ this is obvious since such agent reaches his maximal welfare $u_i = 0$. Pick now $i \in N_+$ and assume $z_i \notin d_i(p, 0)$. Then $d_i(p, 0)$ contains some y_i with $u_i(y_i) > 0$. Let w be a bundle with negative price. Such bundle exists since $p \cdot \omega = \sum_{i\in N} \beta_i = 0$ and $p \neq 0$. Hence the bundle $x_i = y_i + \delta w$ with small enough $\delta > 0$ has negative price $p \cdot x_i < 0$ and $u_i(x_i) > 0$. Contradiction.

Step 2. *If (z, p, β) is a competitive division, then $u(z) = 0$ and (z, p, β') with $\beta' = 0$ is also competitive.*

If $\beta < 0$ we have $u_i(z_i) < 0$ for all $i \in N$. Otherwise $u_i(z_i) \geq 0$ and $p \cdot z_i < 0$ implies as before that $z'_i = 2z_i$ improves U_i (at least weakly) while remaining in $B(p, \beta)$ and lowering i's spending. But $U \in \mathbb{R}^N_=$ is not efficient in a null problem.

Thus $\beta \geq 0$, hence $u_i(z_i) \geq 0$ for all $i \in N$ because the bundle 0 is in the budget set. The problem is null therefore $u(z) = 0$, implying $0 \in d_i(p, \beta)$ and by parsimony (2) $p \cdot z_i \leq 0$, for all i. Hence $z_i \in d_i(p, 0)$ therefore $(z, p, 0)$ is clearly a competitive division.

8.3 An exponential number of competitive divisions

We give an example where $n = 6, m = 5$ where $|CU(\mathcal{P})| = 2^5 - 1 = 31$. Set $N = \{1, 2, 3, 4, 5, 6\}$, $A = \{a, b, c, d, e\}$, and consider the problem

	a	b	c	d	e
u_1	-1	-3	-3	-3	-3
u_2	-3	-1	-3	-3	-3
u_3	-3	-3	-1	-3	-3
u_4	-3	-3	-3	-1	-3
u_5	-3	-3	-3	-3	-1
u_6	-1	-1	-1	-1	-1

There is one symmetric competitive division with uniform price $\frac{6}{5}$ for each bad: it gives $\frac{5}{6}$ of her preferred bad to each agent $1, \cdots, 5$, while agent 6 eats $\frac{1}{6}$ of every bad, precisely his Fair Share. Now for each strict subset of the first five agents, for instance $\{3, 4, 5\}$, there is a competitive allocation where each such agent eats "her" bad in full, while agent 1 shares the other bads with the other agents:

	a	b	c	d	e
z_1	2/3	0	0	0	0
z_2	0	2/3	0	0	0
z_3	0	0	1	0	0
z_4	0	0	0	1	0
z_5	0	0	0	0	1
z_6	1/3	1/3	0	0	0

Here prices are $p = -(\frac{3}{2}, \frac{3}{2}, 1, 1, 1)$.

This construction can be adjusted for each non trivial subset of the first five agents. Then agent 6's utility goes from -1 (his Fair Share) to $-\frac{1}{2}$, when he shares a single bad with a single other agent; utilities of other agents vary also between -1 and $-\frac{1}{2}$.

8.4 Lemma 3

We have $u_i(\omega) < 0$ for every $i \in N$, else the allocation z with $z_i = \omega$ and $z_j = 0$ for $j \neq i$ yields utilities in Γ.

Consider the set of utility profiles dominated by $\mathcal{U} \cap \mathbb{R}^N_-$: $\mathcal{U}_\leq = \{U \in \mathbb{R}^N_- | \exists U' \in \mathcal{U} \cap \mathbb{R}^N_- : U \leq U'\}$. This set is closed and convex and contains all points in $\mathbb{R}^N_-$ that are sufficiently far from the origin. Indeed, any $U \in \mathbb{R}^N_-$ such that $U_N < \min_i u_i(\omega)$, where $U_N = \sum_i U_i$, is dominated by the utility profile $z : z_i = \frac{U_i}{U_N}\omega,\ i \in N$.

Fix $\lambda \geq 0$ and consider the upper contour of the Nash product at λ: $C_\lambda = \{U \in \mathbb{R}^N_- \mid \Pi_N |U_i| \geq \lambda\}$. For sufficiently large λ the closed convex set C_λ is contained in $\mathcal{U}_\leq$. Let λ^* be the minimal λ with this property. Negativity of $\mathcal{P}$ implies that $\mathcal{U}_\leq$ is bounded away from 0 so that λ^* is strictly positive. By definition of λ^* the set C_{λ^*} touches the boundary of $\mathcal{U}_\leq$ at some U^* with strictly negative coordinates. Let H be a hyperplane supporting $\mathcal{U}_\leq$ at U^*. By the construction, this hyperplane also supports C_{λ^*}, therefore U^* is a critical point of the Nash product on $\mathcal{U}_\leq$: that is, U^* maximizes $\sum_{i \in N} \frac{U_i}{|U_i^*|}$ over all $U \in \mathcal{U}_\leq$. So U^* belongs to the Pareto frontier of $\mathcal{U}_\leq$, which is contained in the Pareto frontier of $\mathcal{U}$. Thus U^* is a critical point of the Nash product on $\mathcal{U}$ and belongs to $\mathcal{U}^{eff} \cap \mathbb{R}^N_=$. By the construction any U in the interior of C_{λ^*} is dominated by some $U' \in \mathcal{U} \cap \mathbb{R}^N_-$: so U^* maximizes the Nash product on $\mathcal{U}^{eff} \cap \mathbb{R}^N_-$.

8.5 Lemma 4

In the previous proof note that the supporting hyperplane H to $\mathcal{U}$ at U^* is unique because it is also a supporting hyperplane to C_{λ^*} that is unique. Hence, if $\mathcal{U}$ is a polytope, U^* belongs to a face of maximal dimension.

When utilities are additive, both sets $\mathcal{U}$ and $\mathcal{U}_\leq$ are polytopes. Let $D \subset \mathbb{R}^{N \times A}$ be the set of all u such that the problem (N, A, u) is negative and U^* is not unique. By the above remark if $u \in D$ then for some $\lambda > 0$ the set $\mathcal{U}_\leq$ has at least two faces F and F' of maximal dimension that are tangent to the surface S_λ: $\prod_{i \in N} |U_i| = \lambda$, $U \in \mathbb{R}^N_=$. The condition that F is tangent to S_λ fixes λ. The set of all hyperplanes tangent to a fixed surface S_λ has dimension $|N| - 1$ (for every point on S there is one tangent hyperplane) though the set of all hyperplanes in $\mathbb{R}^N$ is $|N|$-dimensional. Hence tangency of F' and S_λ cuts one dimension. So D is contained in a finite union of algebraic surfaces and, therefore, has Lebesgue-measure zero.

8.6 Proposition 1

Recall the complete proof is in Proposition 4 and Theorem 2 of the companion paper. We fix $N = \{1, 2, 3, 4\}$, $A = \{a, b\}$ and construct two problems. The set $\mathcal{A}(0)$ of efficient and envy-free allocations in the first problem $\mathcal{P}(0)$ has three connected components $\mathcal{B}_i$, $i = 1, 2, 3$. By a symmetry argument it is without loss of generality to assume that an efficient and envy-free rule picks an allocation in $\mathcal{B}_2$ or $\mathcal{B}_3$. Then as we move linearly from $\mathcal{P}(0)$ to $\mathcal{P}(1)$, the components $\mathcal{B}_2$ and $\mathcal{B}_3$ of $\mathcal{A}$ disappear, without ever intersecting $\mathcal{B}_1$, until eventually $\mathcal{A}(1) = \mathcal{B}_1$. This forces a discontinuity in the rule. Define

$$\mathcal{P}(0):\ \begin{array}{ccc} & a & b \\ u_1 & 1 & 5 \\ u_2 & 1 & 4 \\ u_3 & 4 & 1 \\ u_4 & 5 & 1 \end{array} \quad ; \quad \mathcal{P}(1):\ \begin{array}{ccc} & a & b \\ u_1 & 1 & 5 \\ u_2 & 1 & 4 \\ u_3 & 1 & 2 \\ u_4 & 2 & 1 \end{array}$$

If an allocation z is efficient then at most one agent i eats both bads. Suppose it is agent 1: then if z is also envy-free it takes the form

$$z = \begin{array}{ccc} & a & b \\ z_1 & 1 & 1-3x \\ z_2 & 0 & x \\ z_3 & 0 & x \\ z_4 & 0 & x \end{array} \quad \text{where } \frac{3}{10} \leq x \leq \frac{5}{16}$$

This interval in $\mathcal{F}$ is $\mathcal{B}_1$, the first component of $\mathcal{A}(0)$, in which $x = \frac{3}{10}$ yields the competitive allocation z^1. Exchanging the roles of a and b and of agents i and $5-i$ we find symmetrically the component $\mathcal{B}_3$:

$$z = \begin{array}{ccc} & a & b \\ z_1 & x & 0 \\ z_2 & x & 0 \\ z_3 & x & 0 \\ z_4 & 1-3x & 1 \end{array} \quad \text{where } \frac{3}{10} \leq x \leq \frac{5}{16}$$

with another competitive allocation z^2 at $x = \frac{3}{10}$. The third component $\mathcal{B}_2$ is itself symmetric around the symmetric competitive allocation z^3 where agents $1, 2$ split a equally, while $3, 4$ split b. Unlike the two other components, $\mathcal{B}_2$ is of dimension two: half of $\mathcal{B}_2$ is the following set of allocations

$$z = \begin{array}{ccc} & a & b \\ z_1 & x & 0 \\ z_2 & 1-x & 1-2y \\ z_3 & 0 & y \\ z_4 & 0 & y \end{array} \quad \text{where } x + 5y \leq 3 \ ,\ 4x + 3y \leq 5 \ ,\ 2x + 8y \geq 5$$

the other half obtaining by exchanging the roles of a and b and of agents i and $5-i$. So $\mathcal{A}(0) = \cup_{i=1}^{3} \mathcal{B}_i$.

Turning to $\mathcal{P}(1)$, we see that the above allocation z^1 is still competitive, and the corresponding connected component of $\mathcal{A}(1)$ is still the interval $\mathcal{B}_1$. However there is no other competitive allocation: $\mathcal{A}(1) = \mathcal{B}_1$. Finally we consider the interval of problems $\mathcal{P}(\lambda)$ connecting our two problems

$$\mathcal{P}(\lambda): \begin{array}{ccc} & a & b \\ u_1 & 1 & 5 \\ u_2 & 1 & 4 \\ u_3 & 4-3\lambda & 1+\lambda \\ u_4 & 5-3\lambda & 1 \end{array} \quad 0 \leq \lambda \leq 1$$

and check that $\mathcal{B}_1$ remains a component for all λ while $\mathcal{B}_2$ and $\mathcal{B}_3$ shrink and eventually disappear after $\lambda = \frac{3}{4}$.

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