On Time Consistency, Pareto-Optimality [& Equilibrium] in a Model with Heterogeneous Quasi-Hyperbolic Discounting Agents

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A Classical Fable

- Heterogeneous Aesop/La Fontaine's flavoured Grasshoppers and Ants-like Environment with distinct time perceptions.
  1. Short-sighted Grasshoppers are just concerned by today.
  2. Far-Sighted ants think about the future and do save most of their income.
  3. After a succession of generations, the ants would hold the whole wealth while the Grasshoppers would be in the need to reimburse their debts.
- Role for a benevolent social planner?
  1. Is it to follow the ants or the Grasshoppers perception?
  2. Is it to rely on the ants or on the Grasshoppers in order to determine a welfare criterion?
  3. Upon the existence of some irrational behaviour, should a benevolent planner constrain the agents to follow behaviours that differ from their own choice and build some sort of happiness against their own will.

Simple Answers & Problems at stake

- Simple Answers:
  1. Benevolent planner maximises long-run welfare with the utilities of the agent, planner ends up with a greater weight for the positive long-run valuation of the ants.
  2. Even though the grasshoppers are neglected and only the ants are taken into account, sounds like a Pareto-Optimal solution.
- Three issues at stake:
  1. How does a centralized planning solution relate to Time Consistency?
  2. How does a centralized planning solution relate to Pareto-Optimality for the Agents?
  3. [How does a centralised solution relate to the equillibrium solution?]

Some New Counter-Fables (I)

- Environnement where Grasshoppers and ants do co-exist:
  1. The Negishi flavor social optimum as defined by the benevolent planner is not time-consistent.
  2. Would it be stated at another period, it would lead to another solution.
- New solution concept for the social optimum of the benevolent planner.
  1. Pareto Optima constrained to be time-consistent.
  2. A long run where grasshoppers and ants end up having the same consumption.
  3. A long run where the time perception / preference corresponds to an average of the grasshoppers and ants approaches.

Some New Counter-Fables (II)

- Environment where grasshoppers would further assume some sort of irrationality:
  1. Grasshoppers procastinate and delay up to tomorrow their savings decision.
  2. Today grasshoppers disagree with tomorrow ones.
  3. A grasshopper is now considered as a dynasty / sequence of successive incarnations, a given period incarnation being in charge of the consumption and savings decision for that period.

Some New Counter-Fables (II *bis*)

- Two concepts of social optimum are introduced:
  1. A dynasties approach.
     (i) Appropriately weight all of the utilities of the successive incarnations in order to bring them back to the ants preferences.
     (ii) All of the issues that spring from some degree of irrationality or heterogeneity are cirvumvented but the grasshoppers preferences are somewhat forgotten.
  2. An heterogeneity-based approach.
     (i) Pareto Optima are constrained to be time-consistent.
     (ii) A long run where the time perception / preference of the benevolent planner corresponds to an average of the grasshoppers dynasties and ants approaches.
     (iii) A Happy Ending for both grasshoppers and ants, a new epilogue for the fable!

- Given agent $i \in \{1, 2, \ldots, n\}$: *self* at time $t \in \mathbb{N}$ ranks consumption sequences according to:

$$u\left(c_t^i\right) + \beta_i \left[\sum_{\tau=1}^{+\infty} \left(\delta_i\right)^\tau u\left(c_{t+\tau}^i\right)\right], \ \beta_i > 0, \ \delta_i \in]0, 1[. \tag{2}$$

- With a present bias ($\beta_i \neq 1$), decisions of agent $i \in \{1, 2, \ldots, n\}$ stem from the Nash equilibrium of a strategic game between the successive *selves*.
- ASSUMPTION $u(\cdot)$ is defined on $\mathbb{R}_+$, of class $\mathscr{C}^2$, with $\mathrm{D}u(c) > 0$, $\mathrm{D}^2 u(c) < 0$, for $c > 0$ and $\mathrm{D}u(0) = +\infty$, $\mathrm{D}u(+\infty) = 0$, $\delta_i \in]0, 1[$, $i = 1, \ldots, n$ and $\delta_1 > \cdots > \delta_n$.
- Temporally inconsistent preferences (though consistent starting from date $t \geqslant 1$).

The Environment
Extra Restrictions & Ressource Constraint

- Environment with n heterogeneous agents
- $1 > \delta_1 > \delta_2 > \cdots > \delta_n > 0$.
- Agents utilities are weighted according to factors $\lambda_1, \ldots, \lambda_n$ within the Negishi-type objective of the benevolent planner.

- Two essential difficulties:
  1. For $\beta_i \neq 1$, temporal inconsistency of individual preferences and present biased agents.
  2. For $\beta_i = 1$, present unbiased agents but heterogeneity in the parameter δ_i results in a temporal inconsistency in the objective of the Centralized Planner (Drugeon & Wigniolle [JME, 2016]).

The Consistency Issue [$\beta_i = 1$, Two agents case, Equal weights]

- Comparison between two distinct incarnations of the social planner (two agents case and identical weights).
  1. the date 0-incarnation objective function:

$$\sum_{t=0}^{+\infty} \sum_{i=1}^{2} (\delta_i)^t u(c_t^i)$$

  2. the date t-incarnation objective function.
     a/ The inter-temporal utility function of agent i from date t on:

$$\sum_{\tau=0}^{+\infty} (\delta_i)^\tau u(c_{t+\tau}^i).$$

     b/ The objective of date t-incarnation becomes:

$$\sum_{\tau=0}^{+\infty} \sum_{i=1}^{2} (\delta_i)^\tau u(c_{t+\tau}^i).$$

THE CONSISTENCY ISSUE [$\beta_i = 1$, Two agents case]

- Comparison between 1 and 2.b/
  1. Date 0-incarnation of the centralised planner, period t element:

     $$(\delta_1)^t [u(c_t^1) + (\delta_2/\delta_1)^t u(c_t^2)].$$

  2. Date t-incarnation of the social planner, period t element

     $$[u(c_t^1) + u(c_t^2)].$$

- Heterogeneity of discount factors ($\delta_1 \neq \delta_2$) $\Longrightarrow$ objective of the incarnation at date t of the social planner is not consistent with the one of its incarnation at date 0.
- Would the incarnation at date 0 of the social planner be in position to take the whole sequence of decisions from date 0 to $+\infty$, he would select for date t a class of decisions that *differ* from its incarnation at date t.
- Heterogeneity in discount factors hence leads to *temporal inconsistency* in the sequential incarnations of the planner's choices.

Centralized Planning Solution

Definition?

- Exists distinct ways to cope with this twofold temporal inconsistency
  1. $\beta_i \neq 1$
  2. $\delta_1 \neq \delta_2$ for $\beta_i = 1$
- Two main definitions (with many variants and many different properties!) for the Centralized Planner (aiming at Time Consistency):
  1. A Centralized Planning Solution Based upon Dynasties of Selves.
  2. A Planner Incarnation for the Current Selves.

The Dynasties-Based Centralized Planner Solution

Definition for the Single Agent/Dynasty Case

- Agent with preference parameters (β, δ):

$$u\left(c_t\right) + \beta\left[\sum_{\tau=1}^{+\infty}\left(\delta\right)^{\tau} u\left(c_{t+\tau}\right)\right],\ \beta > 0,\ \delta \in]0,1[.$$

- Negishi-like approach over a given dynasty
- Benevolent planner takes a weighted average of the objective functions of the different selves.
- Extra constraint: *Time-consistency* constraint on the optimal programs $\Longrightarrow$ Seeks an exponentially discounted form.

- Weights, denoted as $(\eta_t)_{t\in\mathbb{N}}$, for $\eta_t > 0$ for every $t \in \mathbb{N}$: defined such that the Centralized Planner's criterion corresponds to a weighted average of utilities at any date with an exponential coefficient $\gamma \in]0,1[$.
- Planner considers (I)

$$\sum_{t=0}^{+\infty} \eta_t \left[u(c_t) + \beta \sum_{\tau=1}^{+\infty} u(c_{t+\tau}) \delta^{\tau} \right]$$

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- One looks for a sequence of weights $(\eta_t)_{t\in\mathbb{N}}$ such that the planner's criterion formulates along (II):

$$\eta_0 \sum_{t=0}^{+\infty} \gamma^t u(c_t)$$

The Dynasties-Based Centralized Planner Solution

A First Result for the Single Agent Case

Proposition .1
—*Let* γ *be such that* $\gamma \in [\delta, 1[$. *The objective*

$$\sum_{t=0}^{+\infty} \eta_t \left[u(c_t) + \beta \sum_{\tau=1}^{+\infty} u(c_{t+\tau}) \delta^{\tau} \right]$$

of the centralised planner corresponds to the exponentially discounted form

$$\eta_0 \sum_{t=0}^{+\infty} \gamma^t u(c_t)$$

if and only if the weights η_t *are defined according to*

$$\eta_t = \eta_0 \left\{ \frac{\gamma - \delta}{\gamma - \delta(1-\beta)} \gamma^t + \frac{\beta\delta}{\gamma - \delta(1-\beta)} [\delta(1-\beta)]^t \right\}.$$

The Dynasties-Based Centralized Planner Solution

A Comment for the Single Agent Case

- This proposition generalizes Galperti & Strulovici [2015] where $\gamma = \delta$.
- Values of γ such that $\gamma \in]\delta, 1[$ are also admissible.

Understanding Proposition 1

The Benchmark $\beta_1 = 1$ Configuration

- Selecting weightings η_t^i to the incarnations of the agents i sums up to to modify the discount factor from δ_i to γ for the social planner.
- Clear for $\beta_i = 1$ for which the weightings would indeed simplify to

  $$\eta_t^i = \beta_0^i \cdot \frac{(\gamma - \delta_i)}{\gamma} \gamma^t$$

  for every $t \geqslant 1$.
- Selecting a sum weighed by η_t^i of the successive incarnations of agent i, the planner would indeed transform the agent's discount factor δ_i to a value of γ.
- Proposition 1 allows for recovering temporal consistency for the objective of the social planner but this only be achieved through a *modification* of the preferences of the agents.

Understanding Proposition 1

An Upcounting $\beta_1 > 1$ Configuration

- Under some extra conditions, this result can be extended to a configuration where $\beta_i > 1$.
- To be checked: $\eta_t^i > 0$ keeps on prevailing for every t.
- Argument of the proof of Proposition 1 implies:

$$\eta_1 = (\gamma - \beta\delta)\eta_0.$$

- Restrictions:
  1. $\gamma > \beta\delta$ and thus $\gamma > \delta$. Sufficient for the satisfaction of $\eta_t > 0$ for every t.
  2. Check that, for every t, the inequation:

$$(\gamma - \delta)\gamma^t + \beta\delta[\delta(1-\beta)]^t > 0.$$

     True for even values of t plus corresponds to $(\gamma + \delta\beta - 1)(\gamma - \beta\delta) > 0$ for odd values of t. Also satisfied and establishes the overall property for the odd values of t.

The Fully Consistent Solution

The heterogeneous agents argument

- n agents characterized by pairs (β_i, δ_i), $i = 1, \ldots, n$.

$$u\left(c_t^i\right) + \beta_i \left[\sum_{\tau=1}^{+\infty} \left(\delta_i\right)^\tau u\left(c_{t+\tau}^i\right)\right], \ \beta_i > 0, \ \delta_i \in]0, 1[.$$

- Objective of the social planner defined through a double summation over n agents made of an infinity of selves:

$$\sum_{i=1}^{n} \sum_{t=0}^{+\infty} \eta_t^i \left[u(c_t^i) + \beta_i \sum_{\tau=1}^{+\infty} u(c_{t+\tau}^i)(\delta^i)^\tau \right]$$

PROPOSITION .2

—*Let* γ *be such that* $\gamma \in [\max_{i\in\{1,\dots,n\}}(\delta_i), 1[$. *If the weights* $(\eta_t^i)_{t\in\mathbb{N}}$ *are such that, for every* $i \in \{1,\dots,n\}$,

$$\eta_t^i = \eta_0^i \left\{ \frac{(\gamma-\delta_i)}{\gamma-\delta_i(1-\beta_i)}\gamma^t + \frac{\beta_i\delta_i[\delta_i(1-\beta_i)]^t}{\gamma-\delta_i(1-\beta_i)} \right\},$$

then the objective

$$\sum_{t=0}^{+\infty} \eta_t \left[u(c_t) + \beta \sum_{\tau=1}^{+\infty} u(c_{t+\tau})\delta^\tau \right]$$

of the centralized planner writes down along

$$\sum_{t=0}^{+\infty} \gamma^t \left\{ \sum_{i=1}^{n} \eta_0^i u(c_t^i) \right\},$$

Understanding Proposition 2
Comments (I)

- Proposition 2: brings the objective back to a standard (discounted) centralized one.
- Proposition 1: for agent $i \in \{1, \ldots, n\}$, possible to derive a temporally consistent planner's objective with a coefficient given by $\gamma \geqslant \delta_i$.

Understanding Proposition 2

Comments (II)

- For n agents, temporal consistency is only available if the same value of γ is retained for all of the agents, that limits to the selection of $\gamma \geqslant \max_{i \in \{1,\ldots,n\}}(\delta_i)$.
- Allowing for distinct values of γ_i for every agent would result in a planner's objective of the following form:
- Drugeon & Wigniolle [JME, 2016] have analyzed such an objective and illustrated the involved temporal inconsistency issue.

The Dynasties-Based Centralized Planner Solution

- The maximization program states as:

$$\max \sum_{t=0}^{+\infty} \gamma^t \left\{ \sum_{i=1}^{n} \eta_0^i u(c_t^i) \right\}$$

$$\text{s.t. } K_{t+1} + \sum_{i=1}^{n} c_t^i = F(K_t, 1) + (1-\mu)K_t, \ K_0 \text{ given.}$$

- Standard problem.
- Economy converges towards the modified golden rule K^* in the long run:

$$D_K F(K^*, 1) + (1-\mu) = \frac{1}{\gamma}.$$

- With such a definition for the objective function, solution is:
    1. Pareto optimal for the successive incarnations of the agents.
    2. Temporally consistent.
- Drawbacks:
    1. Rate of time preference of the planner is unrelated to the ones of the agents for $\gamma > \max_{i \in \{1,\dots,n\}}(\delta_i)$.
    2. Rate of time preference of the planner has the one of the most patient agent for $\gamma = \max_{i \in \{1,\dots,n\}}(\delta_i)$.

The Dynasties-Based Centralized Planner Solution

Status of the Pareto Optimal Solution for $\gamma_i = \delta_i$

- Along the specialisation of Galperti & Strulovici to $\gamma_i = \delta_i$, the planner's solution is available as:

$$\sum_{i=1}^{n} \eta_0^i \left\{ \sum_{t=0}^{+\infty} (\delta_i)^t u(c_t^i) \right\}.$$

- Pareto optimal but time inconsistent solution.
- Planner's maximization at a later date: distinct solution.

- Way of circumventing time-inconsistency: Drugeon & Wigniolle [JME, 2016].
    1. *Imposition* of temporal consistency for the choices of the successive incarnations of the social planner.
    2. By assumption: decision rules will be functions that unequivocally depend on the state variable—the capital stock—under consideration:

$$c_t^i = \vartheta_C^i\big(K_t\big).$$

    3. The solution then comes from a strategic game between the successive incarnations of the planner: *the incarnation at date t determines the best resource allocation at t, having taken into account that all future incarnations of the planner will apply the strategy $c_{t+\tau}^i = \vartheta_C^i\big(K_{t+\tau}\big)$.*
    4. This assumption hence ensures *time consistency* in the optimal choices of the successive social planners.

The Dynasties-Based Centralized Planner Solution

An Alternative Time-Consistent Approach

- The solution concept for this game is similar to the one that was used in the literature on time inconsistency of the consumer problem initiated by Phelps and Pollack
- The social planner is therein viewed as successive incarnations and the decision at date t is taken by the t-th incarnation. Phelps and Pollack went on labelling their equilibrium concept as a *Cournot-Nash Equilibrium*.
- Literature on hyperbolic discounting, various names.
- Judd introduces the seemingly most precise terminology for this equilibrium that is referred as a *continuous differentiable Nash equilibrium*.
- The most common name for this solution concept is however the one of *Markov equilibrium*—vid., *e.g*, Harris and Laibson or Krussel and Smith. At times referred in an intuitive way as describing a *sophisticated behaviour*.

- $J^i\left(K_t\right)$: utility over time of agent i from period $t \geqslant 0$ on, with a rule $\vartheta^i_{t+\tau} = \vartheta^i_C\left(K_{t+\tau}\right)$, $\tau \geqslant 0$, starting with a level of the capital stock given by K_t:

$$J^i\left(K_t\right) = \sum_{\tau=0}^{+\infty} \left(\delta_i\right)^\tau u\left[\vartheta^i_C\left(K_{t+\tau}\right)\right]$$

for $\left\{K_{t+\tau}\right\}_{\tau \geqslant 0}$ recursively defined through

$$K_{t+\tau+1} = F\left(K_{t+\tau}, 1\right) + (1-\eta)K_{t+\tau} - \sum_{i=1}^{2} \vartheta^i_C\left(K_{t+\tau}\right)$$

The Dynasties-Based Centralized Planner Solution

Bellman-Like Equations

$$W\left(K_t\right) := \sum_{i=1}^{2} J^i\left(K_t\right),$$

$$W^{\Delta}\left(K_t\right) := \sum_{i=1}^{2} \delta_i J^i\left(K_t\right),$$

- $\Delta := \left(\delta_1, \delta_2\right)$: vector of heterogeneous discount rates.
- Functions $\vartheta_C^i\left(K_t\right)$ then solve:

$$W\left(K_t\right) = \max_{\{c_t^i\}}\left\{\sum_{i=1}^{2} u\left(c_t^i\right) + W^{\Delta}\left(K_{t+1}\right) \quad \text{s.t. } K_{t+1} = F\left(K_t, 1\right) + (1-\eta)K_t - \sum_{i=1}^{2} c_t^i\right.$$

- Remark: for $\delta_1 = \delta_2 = \delta$, one recovers $W^{\Delta}\left(K_t\right) = \delta W\left(K_t\right)$, previous program uncovers a recursive formulation grounded upon the satisfaction of a Bellman Equation.

The Dynasties-Based Centralized Planner Solution

1. Converges towards a modified golden rule where all discount factors come into play:

$$D_K F(K,1) + (1-\mu) = \left\{ \sum_{i=1}^{n} \frac{\delta_i D\vartheta_C^i(K)}{1 - \delta^i \left[D_K F(K,1) + 1 - \mu - \sum_{i=1}^{n} D\vartheta_C^i(K) \right]} \right\}^{-1} \times \left\{ \sum_{i=1}^{n} \frac{D\vartheta_C^i(K)}{1 - \delta^i \left[D_K F(K,1) + 1 - \mu - \sum_{i=1}^{n} D\vartheta_C^i(K) \right]} \right\};$$

2. Ensued solution is only temporally consistent constrained Pareto Optimal.

The Alternative: A Planner Incarnation for the Current Selves

Definition

- Alternative formulation for the objective of the Benevolent Planner.
- Assumed date-t incarnation of the centralized planner maximizes a weighted average of the objectives of the incarnations of the agents at date t.
- Its objective formulates as:

$$\sum_{i=1}^{n} \eta^i u(c_t^i) + \sum_{i=1}^{n} \eta^i \beta^i \sum_{\tau=1}^{+\infty} (\delta_i)^\tau u(c_{t+\tau}^i)$$

A Planner Solution for the Current Selves

- Date-t incarnation of the centralized planner maximizes the objective

$$\sum_{i=1}^{n} \eta^i u(c_t^i) + \sum_{i=1}^{n} \eta^i \beta^i \sum_{\tau=1}^{+\infty} (\delta_i)^\tau u(c_{t+\tau}^i)$$

under the ressource constraint.

- Trajectory converges towards a modified golden rule $\tilde{K}$ such that:

$$D_K F(\tilde{K}, 1) + 1 - \mu = \frac{1}{\bar{\delta}}, \; \bar{\delta} = \max_i(\delta_i)$$

- Most patient agent preferences determines the characteristics of the long run.
- Solution is pareto optimal for the date-t incarnations of the agents.
- Solution is not temporally consistent.

A Planner Incarnation for the Current Selves

Temporally Consistent Solution I

- Conceivable to introduce a temporally consistent solution for the objective

$$\sum_{i=1}^{n} \eta^i u(c_t^i) + \sum_{i=1}^{n} \eta^i \beta^i \sum_{\tau=1}^{+\infty} (\delta_i)^\tau u(c_{t+\tau}^i)$$

- The planner is *constrained* to decision rules that would solely depend upon the state variable K_t, or $c_t^i = \vartheta_C^i(K_t)$ and $K_{t+1} = \vartheta_K(K_t)$.
- Starting from K_0, sequences $(c_t^i)_{t\in\mathbb{N}}$ and $(K_t)_{t\in\mathbb{N}}$ are recursively determined through:

$$K_{t+1} = F(K_t, 1) + (1-\mu)K_t - \sum_{i=1}^{n} c_t^i.$$

- Payoff function at date t:

$$J^i(K_t) = u[\vartheta_C^i(K_t)] + \beta^i \sum_{\tau=1}^{+\infty} u[\vartheta_C^i(K_{t+\tau})](\delta_i)^\tau$$

- Other auxilliary function:

$$\mathscr{J}^i(K_t) = \sum_{\tau=0}^{+\infty} u[\vartheta_C^i(K_{t+\tau})](\delta_i)^\tau$$

- Reformulation of the payoff function:

$$J^i(K_t) = u[\vartheta_C^i(K_t)] + \beta^i \delta_i \mathscr{J}^i(K_{t+1});$$
$$J^i(K_t) = \beta_i \mathscr{J}^i(K_{t+1}) + (1 - \beta_i) u[\vartheta_C^i(K_t)].$$

Welfare functions:

$$W(K_t) = \sum_{i=1}^{n} \eta^i J^i(K_t), \qquad \mathscr{W}(K_t) = \sum_{i=1}^{n} \eta^i \mathscr{J}^i(K_t) \beta_i \delta_i.$$

- Decision rules $\vartheta_C^i(K_t)$ and $\vartheta_K^i(K_t)$ of the social planner are solutions of the following program:

$$W(K_t) = \max \sum_{i=1}^{n} \eta^i u(c_t^i) + \eta^i \beta_i \delta_i \mathscr{W}(K_{t+1})$$

$$\text{s.t. } K_{t+1} = F(K_t, 1) + (1 - \mu)K_t - \sum_{i=1}^{n} c_t^i$$

- For $\zeta = D_K F(K, 1) + 1 - \mu$, long-run modified golden rule is defined from:

$$\sum_{i=1}^{n} (1 - \beta_i) D\vartheta_C^i(K) = \sum_{i=1}^{n} \frac{D\vartheta_C^i(K)[\zeta\beta_i\delta_i - 1]}{1 - \delta_i[\zeta - \sum_{j=1}^{n} D\vartheta_C^j(K)]}$$

- $\beta = 1$, equation assumes a unique solution $\zeta \in]\min_{i\in\{1,\ldots,n\}}(\delta_i), \max_{i\in\{1,\ldots,n\}}(\delta_i)[$
- Left neighourhood of $\beta = 1$: existence of a unique solution.

The Long-Run

Pareto-Optimality, Temporal Consistency or Equilibrium I

1. Temporally consistent and pareto-optimal (identically discounted agents at a rate $\gamma \geqslant \max_{i \in \{1,\ldots,n\}}(\delta_i)$) solution:
   - All of the agents have a positive consumption in the long-run.
   - Identical for equal weights in the planner's objective.
2. Galperti & Strulovici [2005]'s (discounted agents at rates $\gamma_i = \delta_i$) type of solution;
   1. Temporally inconsistent solution: only the agent with the highest δ_i will have positive long-run consumption.
   2. Temporally consistent solution (Drugeon & Wigniolle [2016]'s type): all of the agents have a positive consumption in the long-run, identical for equal weights in the planner's objective.

The Long-Run

Pareto-Optimality, Temporal Consistency or Equilibrium II

1

2

3 Date t-incarnation of the planner's integrates the objectives of the current Selves at date t.

1. Temporally inconsistent solution: only the most patient agent will have positive long-run consumption.
2. Temporally consistent solution (Drugeon & Wigniolle [2016] 's type): all of the agents have a positive consumption in the long-run, identical for equal weights in the planner's objective.

4 Equilibrium solution [logarithmic preferences]

- Only the agent with the smallest λ_i has positive consumption in the long-run, for

$$\lambda_i = \frac{1-\delta_i}{1-\delta_i+\beta_i\delta_i}.$$

Remaining Issues (I)

1. Existence of the Temporally Consistent Pareto Optimal solution (dynamic game fixed point argument): no result yet available in the literature.
2. Understanding of the concepts of the planner solutions (Negishi weights):
   - a/ Clear for β_i (Competitive Equilibrium with Transfers).
   - b/ Unclear for β (Strategic game between the incarnations/shelves).

Remaining Issues (II)

1 For $\beta_i = 1$, temporally consistent solution can be recovered through the following benevolent planner program:

$$\max_{\{c_t^i\}} \sum_{t=0}^{+\infty} \Delta_t \sum_{i=1}^{n} \lambda_i u^i\left(c_t^i\right) \tag{$\mathscr{P}$}$$

$$\text{s.t. } K_{t+1} = F\left(K_t, 1\right) + (1-\eta)K_t - \sum_{i=1}^{n} c_t^i,$$

$$K_0 \text{ given,}$$

with $\left(\Delta_t\right)_{t\in\mathbb{N}}$ a sequence defined by $\Delta_0 = 1$ and $\Delta_{t+1} = \delta_t \Delta_t$ and $(\delta_t)_{t\in\mathbb{N}}$ a sequence such that, for every $t \geqslant 0$, $0 < \delta_n \leqslant \delta_t \leqslant \delta_1 < 1$ and $\lim_{t\to+\infty}\delta_t = \delta$.

2 Extra condition to be satisfied at each date: $\delta_t = DW^{\Delta}\left(K_{t+1}^*\right)/DW\left(K_{t+1}^*\right)$.

3 Status with $\beta_i \neq 1$?