Welfare in multisector models with endogenous product ranges

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Abstract

We study the welfare implications of intrasectoral shocks whose direct effect is welfare-improving in two-sectoral general equilibrium models with entry. We develop a dual necessary and sufficient condition of welfare losses, which occur when the negative intersectoral effect dominates the positive intrasectoral effect. Our approach is flexible enough to study (i) losses from trade liberalization in models of intraindustry trade between symmetric countries under constant or variable markups, (ii) welfare losses from sectoral productivity improvements in multisectoral models with Melitz-type heterogeneity across firms. We show that, when goods producrd by the two sectors are gross complements, welfare gains will always take place. For the case of gross substitutes, we develop a systematic procedure of constructing examples of losses. In particular, we show that, for losses from trade to occur, neither any asymmetries across countries, nor variable markups in either sector are essential. The only source of possible distortions is the interplay between consumers' love for variety of the good trade in which is not liberalized and the adjustment of sectoral budget shares to the new level of trade freeness.

Keywords: trade; non-traded goods; welfare; elasticity of substitution; variable markups.

JEL Classification:

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1 Introduction

Multisectoral general-equilibrium models are widely used in international trade (Matsuyama, 1995, 2009; ****more references****), imperfect market theory (d'Aspremont and Dos Santos Ferreira, 2016; Behrens et al., 2016) and other fields of applied economic theory... ****TBD****

2 Model

The economy involves two sectors, each of them producing a differentiated good. At this stage, we are fully agnostic about the market structures and technologies in each sector, except that we assume them to be compatible with endogeneous entry. Some specific applications are discussed in Section 4.

2.1 Primitives

There is a unit mass of identical consumers endowed with preferences which are *weakly separable* across the differentiated goods (Blackorby et al., 1978; Varian, 1983). Weak separability means that each consumer is endowed with an upper-tier preference $\succsim$ over $\mathbb{R}^2_+$, and subutilities over the sets of varieties produced in each sector.

We assume that lower-tier preferences are homothetic, which implies that each of them can be represented by an ideal price index with standard properties. As for the upper-tier preference$\succsim$, we only assume that it is continuous, monotonic, and strictly convex. As is well-known (Debreu, 1954), in this case $\succsim$ may be represented by a utility function $U(\cdot,\cdot)$, which is continuous, strictly increasing and strictly quasi-concave.

Let $\theta \in \mathbb{R}_+$ be a scalar parameter, shocks in which have a direct impact on one sector (henceforth, "first sector"), but affect only indirectly the other sector (henceforth, "second sector"). Let P and $\mathcal{P}$ denote the price indices associated with, respectively, the first and the second sector. Let $\alpha(P, \mathcal{P})$ stand for the budget share of the first sector as a function of the sectoral price indices $(P, \mathcal{P})$. The *budget-share function* $\alpha(\cdot,\cdot)$ summarizes all the necessary information about the upper-tier preference $\succsim$. Furthermore, we prove in section 3.3.2 that when the goods of two sectors are gross substitutes, $\alpha(P, \mathcal{P})$ can be viewed *as a primitive of the model.*

2.2 Intrasectoral equilibrium

Whatever the market structure in each sector, we can relate to it an approptiate concept of *intrasectoral equilibrium*, i.e. equilibrium computed for a *given* budget distribution $(s,\, 1-s)$, where s is the budget share of the first sector. Depending on the context, the intrasectoral equilibrium may be a monopolistically competitive equilibrium, a Cournot-Nash or Bertrand-Nash equilibrium with entry, or something substantially more general, e.g. an equilibrium with varying competitive toughness a la d'Aspremont et al.(2007) and d'Aspremont and Dos Santos Ferreira (2009, 2016). In each of these cases, we can compute[1] the intrasectoral equilibrium values of the sectoral price indices, $\widetilde{P}(s, \theta)$ and $\widetilde{\mathcal{P}}(1-s)$, and treat them as functions of s and θ.

Such a "reduced-form" approach requires some minimum additional assumptions about how $\widetilde{P}(s, \theta)$ and $\widetilde{\mathcal{P}}(1-s)$ vary with s and θ. Our most preferred set of assumptions is as follows:

A1: the direct effect of a hike in θ on the first sector price index is positive:

$$\frac{\partial \widetilde{P}}{\partial \theta} > 0;$$

A2: the direct effect of an increase in s reduces the first sector price index:

$$\frac{\partial \widetilde{P}}{\partial s} < 0;$$

A3: the direct effect of an increase in $1-s$ reduces the second sector price index:

$$\frac{\partial \widetilde{\mathcal{P}}}{\partial (1-s)} < 0.$$

Assumptions (**A1**) says that, if the budget structure is kept unchanged, a higher value of θ makes the product produced by the first sector more expensive. In other words, the *direct effect* of a reduction in θ is welfare-improving. However, this effect has *intrasectoral* nature in that it neglects how sectoral budget shares respond to the θ-shock. Therefore, one should also keep in mind the *indirect effect* which is due to the adjustment of consumer's expenditure proportions to the new value of θ. The direction of this *intersectoral* effect is a priori ambiguous, and so is its magnitude: it may either reinforce or oppose the welfare-improving direct effect, and either effect may dominate in the latter case.

One familiar example which comes to mind is an increase in the iceberg trade cost in a standard Krugman-type (or Melitz-type) model of intraindustry trade with monopolistic competition under CES preferences. Because the direct effecct of trade barriers is that consumers become poorer in relative terms, one would expect such a shock to generate welfare losses, while the opposite shock, trade liberalization, is typically viewed as a potential source of welfare gains. However, it has long been recognized by the economic profession that trade liberalization may in fact deteriorate the well-being of individuals. The main reason why the case of losses from trade gained so little attention in the literasture compared to that of gains from trade is probably that, as Helpman and Krugman (1985) point out, "****the citation Kris mentioned, saying it is difficult to construct an explicit example of losses from trade****". In this paper, instead of producing

[1] To be precise, we can do this given that the intrasectoral equilibrium exists.

a gallery of casual examples, we develop a systematic procedure of constructing such examples.

The intuition behind (**A2**)-(**A3**) is as follows: when consumers spend more on the good produced by a particular sector, this invites entry of new firms to that sector, fosters competition, and eventually reduces the price level.

Note that assumptions (**A1**)-(**A3**) capture the *direct effects* of changes in θ and s on the price levels. We will be assuming throughout that (**A1**)-(**A3**) hold. Specific examples micro-founding such behavior of the price indices are developed in Section 4.

2.3 Intersectoral equilibrium

We are now equipped to define the equilibrium in the whole economy.

Definition. *The intersectoral equilibrium is a bundle* $(P^*, \mathcal{P}^*, \alpha^*) \in \mathbb{R}^2_+ \times [0,1]$ *satisfying the following conditions*:

$$P^* = \widetilde{P}(\alpha(P^*, \mathcal{P}^*), \theta), \tag{1}$$

$$\mathcal{P}^* = \widetilde{\mathcal{P}}(1 - \alpha(P^*, \mathcal{P}^*)), \tag{2}$$

$$\alpha^* = \alpha(\widetilde{P}(\alpha^*, \theta), \widetilde{\mathcal{P}}(1 - \alpha^*)). \tag{3}$$

Equations (1) – (2) require consistency: the equilibrium price levels $(P^*, \mathcal{P}^*)$ are the intrasectoral-equilibrium price levels computed at the equilibrium budget share distribution $(\alpha^*, 1 - \alpha^*)$. The fixed-point condition (3) guarantees consistency of the budget structure $(\alpha^*, 1-\alpha^*)$ with rational consumer behavior described by the upper-tier preference $\succsim$, or, equivalently, the budget-share function $\alpha(\cdot, \cdot)$.

3 The welfare implications of a lower θ

This section provides two general results. First, when the outputs of the two sectors are *gross complements*, the intersectoral equilibrium is unique and a lower θ implies welfare gains. Second, we develop a systematic procedure of constructing examples when a lower θ leads to welfare losses for the case when goods are *gross substitutes*. In doing so, we use simple duality and integrability considerations.

3.1 Duality and welfare losses

Let $E(P, \mathcal{P}, U)$ be the expenditure funciton associated with the upper-tier utility $U(\cdot, \cdot)$. Welfare losses under a reduction in θ are equivalent to welfare gains under an increase in θ. This, in turn, is equivalent to a reduction in the equilibrium value E^* of the expenditure function in response to changes $\mathrm{d}P^*$ and $\mathrm{d}\mathcal{P}^*$ in the sectoral price levels triggered by an increase in θ. By the Shepard's lemma, it holds when:

$$\frac{\mathrm{d}E^*}{\mathrm{d}\theta} = u^* \frac{\mathrm{d}P^*}{\mathrm{d}\theta} + v^* \frac{\mathrm{d}\mathcal{P}^*}{\mathrm{d}\theta} < 0, \tag{4}$$

where $u^* = \frac{\alpha^*}{P^*}$ and $v^* = \frac{1-\alpha^*}{\mathcal{P}^*}$ are the equilibrium subutility levels in the first and second sector, respectively. Total differentiation of (1) – (2) with respect to θ yields a linear system of equations in $\begin{pmatrix} \mathrm{d}P^*/\mathrm{d}\theta, & \mathrm{d}\mathcal{P}^*/\mathrm{d}\theta \end{pmatrix}^{\mathrm{T}}$. Solving that system, we get (see Appendix A for details):

$$\begin{pmatrix} \frac{\mathrm{d}P^*}{\mathrm{d}\theta} \\ \\ \frac{\mathrm{d}\mathcal{P}^*}{\mathrm{d}\theta} \end{pmatrix} = \frac{\frac{\partial \widetilde{P}}{\partial \theta}}{1 - \left(\frac{\partial \widetilde{P}}{\partial s} \frac{\partial \alpha}{\partial P} + \frac{\partial \widetilde{\mathcal{P}}}{\partial (1-s)} \frac{\partial (1-\alpha)}{\partial \mathcal{P}} \right)} \begin{pmatrix} 1 - \frac{\partial \widetilde{\mathcal{P}}}{\partial (1-s)} \frac{\partial (1-\alpha)}{\partial \mathcal{P}} \\ \\ \frac{\partial \widetilde{\mathcal{P}}}{\partial (1-s)} \frac{\partial (1-\alpha)}{\partial P} \end{pmatrix}, \tag{5}$$

where all derivatives are computed in equilibrium: $s = \alpha = \alpha^*$, $P = P^*$, and $\mathcal{P} = \mathcal{P}^*$. Combining (5) with (4), we get after simplifications (see Appendix B):

$$\frac{\mathrm{d}E^*}{\mathrm{d}\theta} = \frac{\alpha^*}{P^*} \cdot \frac{\partial \widetilde{P}}{\partial \theta} \cdot \mathcal{R}, \tag{6}$$

where $\mathcal{R}$ is defined by

$$\mathcal{R} \equiv \frac{1 - \mathcal{E}_{1-s}(\widetilde{\mathcal{P}}) \left(\mathcal{E}_{\mathcal{P}}(1-\alpha) + \mathcal{E}_{P}(\alpha) \right)}{1 - \left(\mathcal{E}_{\mathcal{P}}(1-\alpha) \mathcal{E}_{1-s}(\widetilde{\mathcal{P}}) + \mathcal{E}_{s}(\widetilde{P}) \mathcal{E}_{P}(\alpha) \right)}. \tag{7}$$

Due to (**A1**), we have

$$\operatorname{sign}\left(\frac{\mathrm{d}E^*}{\mathrm{d}\theta} \right) = \operatorname{sign}(\mathcal{R}). \tag{8}$$

In other words, the sign of $\mathcal{R}$ is a *sufficient statistic* for determining whether a shock in θ gives rise to welfare gains or welfare losses.

3.2 Gross complementarity across sectors

Consider first the case when goods produced by the two sectors are gross complements.

Proposition 1. *Assume that* (**A1**)-(**A3**) *hold, and that the upper-tier preference* $\succsim$ *is such that the goods produces by the two sectors are gross complements. Then,* (i) *the equilibrium is always interior and unique, and* (ii) *a lower* θ *always yields welfare gains.*

Proof. By definition of gross complements, we have:

$$\mathcal{E}_P(1-\alpha) < 0, \qquad \mathcal{E}_{\mathcal{P}}(\alpha) < 0, \text{ for all } (P, \mathcal{P}) \in \mathbb{R}_+^2,$$

which is equivalent to

$$\mathcal{E}_P(\alpha) > 0, \quad \mathcal{E}_{\mathcal{P}}(1-\alpha) > 0, \text{ for all } (P,\, \mathcal{P}) \in \mathbb{R}^2_+. \tag{9}$$

To prove part (i), denote by $g(\alpha)$ the right-hand side (3) as a function of α. The slope of $g(\alpha)$ is given by

$$g'(\alpha) = \mathcal{E}_s(\widetilde{P})\mathcal{E}_P(\alpha) + \mathcal{E}_{1-s}(\widetilde{\mathcal{P}})\mathcal{E}_{\mathcal{P}}(1-\alpha), \tag{10}$$

where $s = \alpha$. Combining (10) with (9), (**A2**), and (**A3**) implies $g'(\alpha) < 0$ for all $\alpha \in (0,\, 1)$. As a consequence, $g(\cdot)$ is a continuous strictly decreasing function which maps the segment $[0,\, 1]$ into itself. Hence, by the intermediate value theorem, it must have a unique fixed point α^* over $[0,\, 1]$, and this fixed point is interior.

To prove part (ii), note that by (9), the denominator in (7) is positive. Furthermore, using (**A3**) and (9), we find that the numerator in (7) is also positive, whence $\mathcal{R} > 0$. Combining this with (8) implies that $\mathrm{d}E^*/\mathrm{d}\theta > 0$, which violates the condition (4) of losses. Hence, under gross complementarity across sectors, a lower θ always results in welfare improvement. □

3.3 Gross substitutability across sectors

Assume now that goods produced by the two sectors are gross substitutes. Can a reduction in θ lead to welfare losses if the intersectoral equilibrium α^* is unique and interior?

3.3.1 Uniqueness condition

We show in Appendix C that an interior intersectoral equilibrium α^* is always unique if and only if the following inequality holds at any intersectoral equilibrium:

$$\mathcal{E}_s(\widetilde{P})\mathcal{E}_P(\alpha) + \mathcal{E}_{1-s}(\widetilde{\mathcal{P}})\mathcal{E}_{\mathcal{P}}(1-\alpha) < 1. \tag{11}$$

As implied by the fix-point condition (3), the inequality (11) amounts to saying that the budget share function intersects the 45° line only from above. Losses arise if and only if the sufficient statistic $\mathcal{R}$ given by (7) is negative, i.e. when the following inequality holds in equilibrium:

$$\frac{\mathcal{E}_s(\widetilde{P}) - \mathcal{E}_{1-s}(\widetilde{\mathcal{P}})}{1 - \Big(\mathcal{E}_{\mathcal{P}}(1-\alpha)\mathcal{E}_{1-s}(\widetilde{\mathcal{P}}) + \mathcal{E}_s(\widetilde{P})\mathcal{E}_P(\alpha)\Big)} > -\frac{1}{\mathcal{E}_P(\alpha)}. \tag{12}$$

The intuition behind (12) is as follows: welfare losses may occur when (i) love for variety in the second sector is higher than in the first sector, or (ii) the budget share $\alpha(P,\, \mathcal{P})$ is sufficiently responsive to changes in the price indices, so that $g'(\alpha^*)$ is sufficiently close to 1.

3.3.2 The budget share as a primitive.

The budget-share function $\alpha(P, \mathcal{P})$ can actually be viewed *as a primitive of the model.* To be precise, the following result holds.

Lemma. *Any differentiable function $\alpha(P, \mathcal{P})$, such that it (i) decreases in P, (ii) increases in $\mathcal{P}$, and (iii) maps all price vectors $(P, \mathcal{P})$ into $[0, 1]$, is a budget share generated by some monotonic, continuous, and strictly quasi-concave utility function over $\mathbb{R}^2_+$.*

Proof. See Appendix D. □

Due to the Lemma, we can describe the upper-tier preference $\succsim$ using any function $\alpha(P, \mathcal{P})$ satisfying the assumptions of the Lemma.

3.4 Losses from welfare-improving intersectoral shocks

We are now equipped to construct examples of welfare losses under a lower θ.

Proposition 2. *Assume that $\theta \in [\underline{\theta}, \overline{\theta}]$, where $0 < \underline{\theta} < \overline{\theta} < \infty$, while the sectoral price indices are given by*

$$\widetilde{P}(s, \theta) = \theta s^{-a}, \qquad \widetilde{\mathcal{P}}(1-s) = (1-s)^{-b}, \tag{13}$$

where $b > a > 0$. Then, for any size of the gap between a and b, there exist a value $\theta_0 \in (\underline{\theta}, \overline{\theta})$ of θ and an upper-tier utility $U(\cdot, \cdot)$, such that, in the vicinity of $\theta = \theta_0$, (i) the equilibrium is unique and interior, and (ii) a small reduction in θ yields welfare losses.

Proof. See Appendix E. □

Sectoral price indices described by mathematical expressions of the form (13) arise in both the two-sector CES model of trade with symmetric countries (Section 4.1) and the two-sector closed economy model with firm heterogeneity a lá Melitz (Section 4.2). Hence, Proposition 2 leads to examples of both losses from trade and losses from sectoral productivity improvements.

4 Applications

4.1 Losses from trade under asymmetric trade liberalization

TBD

4.2 Losses from intrasectoral productivity improvements

TBD

4.3 ???

TBD

5 Concluding remarks

We have examples of welfare losses from trade in a model which has the following features:

- there are only two countries $\Rightarrow$ no third-country effects are at work;
- countries are symmetric $\Rightarrow$ no potential distortions stemming from differences in country sizes;
- the market structure in both sectors is monopolistic competition $\Rightarrow$ no distortive effects of strategic interaction;
- firms are homogeneous in productivities $\Rightarrow$ no potentially distortive effects of selection and sorting;
- CES subutilities in both sectors $\Rightarrow$ no over/under-entry due to variable markups;
- the intrasectoral elasticities of substitution can be arbitrarily close to each other $\Rightarrow$ no substantial asymmetries in the intrasectoral demand structure;
- a unique equilibrium $\Rightarrow$ no room for qualifying equilibria where losses from trade occur as "bad" equilibria

Therefore, the only two things which may be viewed as a source of distortions in our model are (i) product differentiation, and (ii) the presence of non-traded goods.

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Appendix

A: Proof of (5). Applying the implicit function theorem to (1) – (2), we get:

$$\begin{pmatrix} 1-\frac{\partial \widetilde{P}}{\partial \alpha}\frac{\partial \alpha}{\partial P} & -\frac{\partial \widetilde{P}}{\partial \alpha}\frac{\partial \alpha}{\partial \mathcal{P}} \\ -\frac{\partial \widetilde{\mathcal{P}}}{\partial (1-\alpha)}\frac{\partial (1-\alpha)}{\partial P} & 1-\frac{\partial \widetilde{\mathcal{P}}}{\partial (1-\alpha)}\frac{\partial (1-\alpha)}{\partial \mathcal{P}} \end{pmatrix}\begin{pmatrix} \frac{\mathrm{d}P^*}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\mathcal{P}^*}{\mathrm{d}\tau} \end{pmatrix} = \begin{pmatrix} \frac{\partial \widetilde{P}}{\partial \tau} \\ 0 \end{pmatrix}.$$

Solving this linear system for $\begin{pmatrix} \mathrm{d}P^*/\mathrm{d}\tau, & \mathrm{d}\mathcal{P}^*/\mathrm{d}\tau \end{pmatrix}^{\mathrm{T}}$, we get:

$$\begin{pmatrix} \frac{\mathrm{d}P^*}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\mathcal{P}^*}{\mathrm{d}\tau} \end{pmatrix} = \begin{pmatrix} 1-\frac{\partial \widetilde{P}}{\partial \alpha}\frac{\partial \alpha}{\partial P} & -\frac{\partial \widetilde{P}}{\partial \alpha}\frac{\partial \alpha}{\partial \mathcal{P}} \\ -\frac{\partial \widetilde{\mathcal{P}}}{\partial (1-\alpha)}\frac{\partial (1-\alpha)}{\partial P} & 1-\frac{\partial \widetilde{\mathcal{P}}}{\partial (1-\alpha)}\frac{\partial (1-\alpha)}{\partial \mathcal{P}} \end{pmatrix}^{-1}\begin{pmatrix} \frac{\partial \widetilde{P}}{\partial \tau} \\ 0 \end{pmatrix}.$$

Inverting the matrix yields (5). □

B: **Proof of the decomposition (6)**. Recall that, by homotheticity of the lower-tier utilities, we have: $u = \alpha/P$, $v = (1-\alpha)/\mathcal{P}$. Using this, and plugging the expressions (5) for $\mathrm{d}P^*/\mathrm{d}\tau$ and $\mathrm{d}\mathcal{P}^*/\mathrm{d}\tau$ into (4) and , we get

$$\frac{\mathrm{d}E^*}{\mathrm{d}\tau} = \frac{\partial \widetilde{P}}{\partial \tau}\frac{\frac{\alpha}{P}\left(1-\frac{\partial \widetilde{\mathcal{P}}}{\partial (1-\alpha)}\frac{\partial (1-\alpha)}{\partial \mathcal{P}}\right)+\frac{1-\alpha}{\mathcal{P}}\frac{\partial \widetilde{\mathcal{P}}}{\partial (1-\alpha)}\frac{\partial (1-\alpha)}{\partial P}}{1-\left(\frac{\partial \widetilde{P}}{\partial \alpha}\frac{\partial \alpha}{\partial P}+\frac{\partial \widetilde{\mathcal{P}}}{\partial (1-\alpha)}\frac{\partial (1-\alpha)}{\partial \mathcal{P}}\right)}. \tag{14}$$

Simplifying the numerator of the fraction in the right-hand side of (14) yields:

$$\frac{\alpha}{P}\left(1-\frac{\partial \widetilde{\mathcal{P}}}{\partial (1-\alpha)}\frac{\partial (1-\alpha)}{\partial \mathcal{P}}\right)+\frac{1-\alpha}{\mathcal{P}}\frac{\partial \widetilde{\mathcal{P}}}{\partial (1-\alpha)}\frac{\partial (1-\alpha)}{\partial P}$$

$$= \frac{\alpha}{P}\left(1 - \frac{\partial\widetilde{\mathcal{P}}}{\partial(1-\alpha)}\frac{\partial(1-\alpha)}{\partial\mathcal{P}} + \frac{1-\alpha}{\mathcal{P}}\frac{\partial\widetilde{\mathcal{P}}}{\partial(1-\alpha)}\frac{\partial(1-\alpha)}{\partial P}\frac{P}{\alpha}\right)$$

$$= \frac{\alpha}{P}\left(1 - \frac{1-\alpha}{\widetilde{\mathcal{P}}}\frac{\partial\widetilde{\mathcal{P}}}{\partial(1-\alpha)}\frac{\partial(1-\alpha)}{\partial\mathcal{P}}\frac{\widetilde{\mathcal{P}}}{1-\alpha} + \frac{1-\alpha}{\mathcal{P}}\frac{\partial\widetilde{\mathcal{P}}}{\partial(1-\alpha)}\frac{\partial(1-\alpha)}{\partial P}\frac{P}{1-\alpha}\frac{1-\alpha}{\alpha}\right)$$

$$= \frac{\alpha}{P}\left(1 - \mathcal{E}_{\mathcal{P}}(1-\alpha)\mathcal{E}_{1-\alpha}(\widetilde{\mathcal{P}}) + \mathcal{E}_{1-\alpha}(\widetilde{\mathcal{P}})\mathcal{E}_{P}(1-\alpha)\frac{1-\alpha}{\alpha}\right)$$

$$= \frac{\alpha}{P}\left(1 - \mathcal{E}_{\mathcal{P}}(1-\alpha)\mathcal{E}_{1-\alpha}(\widetilde{\mathcal{P}}) - \mathcal{E}_{1-\alpha}(\widetilde{\mathcal{P}})\mathcal{E}_{P}(\alpha)\right).$$

Note also that in equilibrium we have

$$\frac{\partial\widetilde{P}}{\partial\alpha}\frac{\partial\alpha}{\partial P} = \mathcal{E}_{\alpha}(\widetilde{P})\mathcal{E}_{P}(\alpha), \qquad \frac{\partial\widetilde{\mathcal{P}}}{\partial(1-\alpha)}\frac{\partial(1-\alpha)}{\partial\mathcal{P}} = \mathcal{E}_{\mathcal{P}}(1-\alpha)\mathcal{E}_{1-\alpha}(\widetilde{\mathcal{P}}),$$

whence the denominator in (14) becomes

$$1 - \frac{\partial\widetilde{P}}{\partial\alpha}\frac{\partial\alpha}{\partial P} - \frac{\partial\widetilde{\mathcal{P}}}{\partial(1-\alpha)}\frac{\partial(1-\alpha)}{\partial\mathcal{P}} = 1 - \mathcal{E}_{\mathcal{P}}(1-\alpha)\mathcal{E}_{1-\alpha}(\widetilde{\mathcal{P}}) - \mathcal{E}_{1-\alpha}(\widetilde{\mathcal{P}})\mathcal{E}_{P}(\alpha).$$

Plugging everything into (14) completes the proof. □

C: Proof of (11). Assume there exists an equilibrium α^* where (11) is violated, then, using (10), we find the right-hand side of (3) intersects the 45°-line from below: $g'(\alpha^*) > 1$. In this case, another equilibrium $\alpha^{**} > \alpha^*$ exists. Indeed, because $g'(\alpha^*) > 1$, there must exist a small $\varepsilon > 0$, such that $g(\alpha^* + \varepsilon) > \alpha^* + \varepsilon$. Hence, $g(\alpha)$ maps $[\alpha^* + \varepsilon, 1]$ into itself. . Hence, by the intermediate value theorem, a fixed point $\alpha^{**} \in [\alpha^* + \varepsilon, 1]$ exists. □

D: Proof of the Lemma. It suffices to prove that the demand functions

$$u(y, P, \mathcal{P}) \equiv \frac{y}{P}\alpha\left(\frac{P}{y}, \frac{\mathcal{P}}{y}\right), \quad v(y, P, \mathcal{P}) \equiv \frac{y}{\mathcal{P}}\left(1 - \alpha\left(\frac{P}{y}, \frac{\mathcal{P}}{y}\right)\right), \tag{15}$$

where $y > 0$ is consumer's income, satisfy the following properties:

(i) budget-balancedness: for all

$$Pu(y, P, \mathcal{P}) + \mathcal{P}v(y, P, \mathcal{P}) = y;$$

(ii) the Slutsky matrix

$$\mathbf{S}(P,\mathcal{P},y) \equiv \begin{pmatrix} \frac{\partial u}{\partial P} + u\frac{\partial u}{\partial y} & \frac{\partial u}{\partial \mathcal{P}} + v\frac{\partial u}{\partial y} \\ \\ \frac{\partial v}{\partial P} + u\frac{\partial v}{\partial y} & \frac{\partial v}{\partial \mathcal{P}} + v\frac{\partial v}{\partial y} \end{pmatrix}.$$

of the demand system (15) is *symmetric and negative semidefinite.*

By Antonelli's (1886) integrability theorem, equation (15) describes true Marshallian demands generated by some continuous, monotonic and strictly quasi-concave utility if and only if (i) and (ii) hold.

Budget balancedness ensues immediately from (15). Moreover, the demands (15) are homogeneous of degree zero in $(P,\mathcal{P},y)$. For the case of two goods, this is sufficient for $\mathbf{S}(P,\mathcal{P},y)$ to be symmetric (see, e.g., Jehle and Reny, 2011, Ch. 2).

To prove that $\mathbf{S}(P,\mathcal{P},y)$ is negative semidefinite, observe that the price vector $\mathbf{p} \equiv (P,\mathcal{P})$ annihilates the Slutsky matrix due to the budget balancedness. Furthermore, the vector $\mathbf{e} \equiv (1,0)$ always renders the quadratic form induced by negative. Indeed, the $(1,1)$-entry of the Slutsky matrix is given by

$$s_{11} \equiv \frac{\partial u}{\partial P} + u\frac{\partial u}{\partial y} = -(1-\alpha)\left(\frac{y\alpha}{P^2} - \frac{1}{P}\frac{\partial \alpha}{\partial (P/y)}\right) - \frac{\mathcal{P}}{P^2}\alpha\frac{\partial \alpha}{\partial (\mathcal{P}/y)} < 0,$$

so we get

$$\mathbf{e}^T\mathbf{S}\mathbf{e} = s_{11} < 0. \tag{16}$$

Because the vectors $\mathbf{e} = (1,0)$ and $\mathbf{p} = (P,\mathcal{P}) \in \mathbb{R}^2_{++}$ form a basis in $\mathbb{R}^2$, for any vector $\mathbf{h} = (h_1,h_2) \in \mathbb{R}^2$ the coefficients $\theta_1,\theta_2 \in \mathbb{R}$ must exist, such that

$$\mathbf{h} = \theta_1\mathbf{e} + \theta_2\mathbf{p}.$$

Computing $\mathbf{h}^T\mathbf{S}\mathbf{h}$, we get:

$$\mathbf{h}^T\mathbf{S}\mathbf{h} = \theta_1^2\mathbf{e}^T\mathbf{S}\mathbf{e} + 2\theta_1\theta_2\mathbf{e}^T\mathbf{S}\mathbf{p} + \theta_1^2\mathbf{p}^T\mathbf{S}\mathbf{p} = \theta_1^2\mathbf{e}^T\mathbf{S}\mathbf{e} = \theta_1^2 s_{11}.$$

Due to (16), we always have $\theta_1^2 s_{11} \leq 0$, whence $\mathbf{S}(P,\mathcal{P},y)$ is negative semidefinite.□

E: Proof of Proposition 2. Consider the following budget-share function:

$$\alpha(P,\mathcal{P}) = \frac{\varepsilon}{2} + (1-\varepsilon)\frac{\mathcal{P}^{1/b}}{kP^{1/a} + \mathcal{P}^{1/b}}, \tag{17}$$

where ε and k satisfy the following restrictions:

$$0 < \varepsilon < 1, \qquad \left(\overline{\theta}\right)^{-1/a} < k < \left(\underline{\theta}\right)^{-1/a}. \tag{18}$$

By the Lemma, there exists a continuous, strictly monotonic and strictly quasi-concave upper-tier utility $U(u, v)$ over $\mathbb{R}_+^2$, such that $\alpha(P, \mathcal{P})$ given by (17) is the budget share of the first sector induced by $U(u, v)$.

Combining (17) with (13), we find that the fixed-point condition (3) becomes

$$\alpha^* = \frac{\varepsilon}{2} + (1-\varepsilon)\frac{\alpha^*}{(1-k\theta^{1/a})\alpha^* + k\theta^{1/a}}. \tag{19}$$

Now choose θ_0 so that is satisfies $k\theta_0^{1/a} = 1$. That $\theta_0 \in (\underline{\theta}, \overline{\theta})$ is guaranteed by (18). Setting $\theta = \theta_0$ in (19) yields:

$$\alpha^* = \frac{\varepsilon}{2} + (1-\varepsilon)\alpha^*.$$

Hence, the intersectoral equilibrium is obviously unique, interior, and the following equalities hold:

$$\alpha^* = \frac{1}{2}, \qquad g'(\alpha^*) = 1 - \varepsilon. \tag{20}$$

This proves part (i). To prove part (ii), observe that the elasticities of sectoral price indices are given by

$$\mathcal{E}_s(\widetilde{P}) = -a, \qquad \mathcal{E}_{1-s}(\widetilde{\mathcal{P}}) = -b.$$

Furthermore, using (20), the condition (12) can be cast as follows:

$$b - a > -\frac{\varepsilon}{\mathcal{E}_P(\alpha)}. \tag{21}$$

It remains to evaluate the equilibrium value of $\mathcal{E}_P(\alpha)$. Using (17), we obtain

$$kP^{1/a}\mathcal{P}^{-1/b} = \frac{1-\alpha-\varepsilon/2}{\alpha-\varepsilon/2},$$

which entails

$$\mathcal{E}_P(\alpha) = -\frac{(\alpha-\varepsilon/2)\,(1-\alpha-\varepsilon/2)}{(1-\varepsilon)\alpha}.$$

Since $\alpha^* = 1/2$, evaluating $\mathcal{E}_P(\alpha)$ in equilibrium yields

$$\mathcal{E}_P(\alpha) = \frac{\varepsilon - 1}{2}. \tag{22}$$

Plugging (22) into (21), we come to the following representation of (12):

$$b - a > \frac{2\varepsilon}{1-\varepsilon}. \tag{23}$$

Because $b > a > 0$, the right-hand side of (23) is strictly positive. Hence, (23) holds true under sufficiently small values of ε. This completes the proof. □