The merits of privileged parking

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Abstract

The paper investigates under what conditions it is optimal to exclude some motorists searching for parking from occupying a vacant parking bay. Privileged parking is found to be optimal if motorists are heterogenous and can steer their search towards or away from such parking. The socially optimal allocation of privileges and search strategies are characterized. The second-best pricing policies in the presence of cruising-for-parking externality are described; short-term parkers should always be allowed to take the first vacant bay they find. A model extension studies technologically modified "special needs" parking. Unlike existing policies that make such parking exclusive for special-needs motorists, the optimal policy makes it available for an extra fee to anyone, while increasing the number of special-needs bays so that even the initial users are better off.

Keywords: privileged parking, loading zones, congestion pricing, cruising for parking

JEL codes: H42, R41, R48

1. Introduction

Many governments practice discriminatory policy for geographically homogenous curbside parking. A large number of municipalities around the world have dedicated "loading zones". Most municipalities in developed countries have exclusive parking for the disabled, with or without special access design. With the advent of electric vehicles (EV hence-

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forth) that require lengthy periods of battery charging, cities introduce exclusive EV parking equipped with charging infrastructure.

Policies regarding privileged parking vary across countries and cities, but typically aim to exclude the majority of motorists searching for parking (*searchers* henceforth) in favor of certain special groups. For example loading zones in Melbourne, Australia can be used only by commercial vehicles such as delivery trucks, buses, and taxis; private passenger cars are not allowed to stop there even briefly. The "yellow zones" in San Francisco, California have similar regulations, while the "white zones" do allow any vehicle to stop for passenger unloading, as long as the parking session does not exceed 5 minutes. Handicapped, EV parking, and taxi stands are typically reserved for the corresponding types of vehicles, excluding all others.

This paper aims to understand theoretically the merits of privileged parking. To fix ideas, we will say that a parking bay is *privileged* if some kind of government regulation prevents some searchers from parking at that bay. The questions investigated in the paper include

- Can it be optimal to segregate technologically and geographically identical parking by introducing privileges?
- Should privilege allocation be horizontal (some bays are exclusive for searcher A while others are exclusive for B) or vertical (some bays are exclusive for A while others are available to both A and B)?
- What policy can implement, precisely or approximately, socially optimal allocation of parking?
- How do changes in overall demand for parking (e.g. on weekends) affect optimal privilege policy?
- If some searchers require technologically modified bays (accessible, having a charging

station, etc.), how should access of other searchers to such bays be regulated? And what is the socially optimal amount of such parking supply?

To answer these questions, this paper formulates a dynamic model of parking demand and search in a geographically homogenous area with searchers that differ in desired parking duration and the value of search time. Throughout most of the paper, any parking bay is technologically accessible by any searcher. All parking supply is exogenously divided into two types to which different policies can be applied; the paper focuses on whether such heterogeneity of policies is useful, and how exactly the policies should differ. In section 7, some searchers are *special needs*, i.e. require specially designed parking infrastructure. The amount of special needs parking supply is endogenized.

To the best of my knowledge, there is no existing scientific analysis of policy discrimination in otherwise homogenous parking. There exist papers studying interaction between parking policy of different suppliers, e.g. municipally-owned curbside parking and privately owned garage parking, as in Arnott (2006), Inci and Lindsey (2015), or in Gragera and Albalate (2016). Another line of research is policy heterogeneity caused by spatial inequality of parking demand, as in Anderson and de Palma (2004). Inci and Lindsey (2015) have a model with heterogenous searchers and allow for different per-hour parking fees for different types of searchers, but in their setting the socially optimal per-hour fees are the same for all.

Zakharenko (2019) in section 4.1 mentions that if all parking supply can be separated into two isolated but geographically identical areas, and if searchers are heterogenous, it might be optimal to set different prices per hour in the two areas. This observation however is not the main focus of that paper and is not studied in detail. Moreover, the paper assumes that searchers can fully control which of the two parking types they will inspect for parking, therefore they will never pass by a bay from which they can be excluded by regulation. The current paper offers a more general model where searchers sometimes have to pass by vacant bays at which they cannot park, which is empirically relevant.

Section 7 of the current paper is also probably the first theoretical study of the economics of special-needs parking. In particular, there is no known research on how the accessible parking should optimally be supplied. Given the fact that such parking constitutes about 10% of all parking supply in developed countries, and plays an important role in everyday life of people with disabilities, its socially optimal provision is an important question of urban economics.

2. Model

2.1. Parking supply

Consider a continuous-time environment with a continuum of parking of exogenous amount normalized to unity. All parking is divided into two types, which we will denote 1 and 2. All parking bays of all types are geographically equally attractive to all motorists. Throughout most of the paper, the two types of parking are technologically identical (i.e. any searcher can use bays of both types), and the share S_i of type-i parking is exogenously given. Such exogenous division of parking can be driven, for example, by the street pattern of the area in question. Section 7 studies the alternative setting where type-2 parking is "special needs", i.e. is the only option for some searchers; the share of such parking is endogenized.

2.2. Parking demand

The motorists searching for parking differ in their desired parking duration t and in cost c of parking search time; the pair $\{c, t\}$ is referred to as the searcher's *type*. The desired parking duration can range from zero to infinity; for simplicity, we assume that it is exogenous for each searcher and does not respond to policy or aggregate equilibrium conditions. The search cost c may include

1. private value of time spent on search *per se*;

2. value of time spent on roundtrip walk between parking location and final destination;[2]

3. "cruising for parking" external effects, including delays of other traffic, pollution, and accident risks, caused by the searching vehicle.

The last, external, component of the search cost is assumed to be the same for all, while the first two may vary from zero to infinity. In particular, searchers with disabilities are assumed to have high walking component of the search cost.

The inflow of searchers of type $\{c, t\}$ that initiate the search during a representative unit of time is $a(c, t)$. To keep the model simple, we assume that such inflow is exogenous. It is also assumed to be time-invariant, which implies that all equilibrium aggregate parameters will be time-invariant, as well. Because all arriving searchers will end up being parked, the outflow of motorists of type $\{c, t\}$ from the search mode into the parked mode is $a(c, t)$, as well. Thus, the total mass of motorists parked at any time is $D \equiv \int_{c,t} a(c, t) t \mathrm{d}c \mathrm{d}t$, which should be strictly less than parking supply for any vacancy to exist:

$$D < 1, \tag{1}$$

Denote by q_i the *occupancy*, i.e. the proportion of occupied bays of type i; the average across both types of parking occupancy must be equal to D,

$$q_1 S_1 + q_2 S_2 = D. \tag{2}$$

We also impose the following regularity condition on the searcher type distribution.

Assumption 1. *For any $k_1 \geq 0$ and any $k_2 > k_1$, the quantity $\int_{t=0}^{\infty} \int_{c=k_1 t}^{k_2 t} a(c, t) \mathrm{d}c \mathrm{d}t$ is positive and differentiable with respect to k_1, k_2.*

[2] Assuming the distance of such walk increases in proportion with the search time, every extra minute of search entails a constant walking cost.

2.3. Search technology

We use a classical model of parking search, nicknamed the "binomial approximation" by Arnott and Williams (2017), with some modifications. We will assume that parkers randomly sample one parking bay at a time, such that every new try is independent from previous ones. Time is normalized so that one bay per unit of time is sampled. While the occupancy status of the drawn parking bay is completely random, searchers can somewhat influence the type of parking (1 or 2) that they draw. We will nickname such model of search by *imperfect steering.* The term *steering* is motivated by the idea that searchers, by optimizing their search route, can increase the fraction of their most desired type of parking that they encounter during the search. Steering is *imperfect* because, despite their best effort, the searchers sometimes still encounter the type of parking they do not desire or from which they are excluded.

Specifically, the searchers are able to choose the probability of drawing type-1 parking from the range $[\underline{s}, \bar{s}]$ such that $\underline{s} \leq S_1 \leq \bar{s}$. Lack of choice $\underline{s} = S_1 = \bar{s}$ implies that they have no control over the search process; section 3.4 shows that in this case, privileged parking is not optimal. In contrast, the case $\underline{s} = 0, \bar{s} = 1$ corresponds to full control of the search process and excludes the chance of drawing an undesired type of parking. By $s_1(c,t) \in [\underline{s}, \bar{s}]$ we will denote the steering strategy of a type $\{c,t\}$ searcher, i.e. their chosen probability of drawing type-1 parking. Similarly, $s_2(c,t) = 1 - s_1(c,t)$ is the probability of the alternative.

If the drawn parking bay is occupied, search must continue. If the drawn bay of type $i \in \{1,2\}$ is vacant, the searcher with parameters $\{c,t\}$ parks there with probability $p_i(c,t)$ which is driven by the government policy introduced below. We will nickname $p_i(c,t)$ as the *parking strategy* of searcher $\{c,t\}$ at vacant bay of type i.

Given these assumptions, the *chance of success* of a searcher of type $\{c,t\}$, i.e. the probability that she ends up being parked after each draw, is

$$r(c,t) = \sum_{i=\{1,2\}} s_i(c,t)(1-q_i)p_i(c,t). \tag{3}$$

2.4. Steady state occupancy

Suppose the proportion of vacant bays of type i that become occupied during an infinitesimal time period $\mathrm{d}\tau$ is $E_i\mathrm{d}\tau$; the proportion of occupied bays that are being vacated during the same period is $X_i\mathrm{d}\tau$. In the steady state, the total mass of bays being occupied, $(1-q_i)E_i\mathrm{d}\tau$ must equal the total mass of bays being vacated, $q_iX_i\mathrm{d}\tau$, hence

$$q_i = \frac{E_i}{E_i + X_i} = \frac{T_i}{1+T_i}, \tag{4}$$

where $T_i \equiv \frac{E_i}{X_i}$.

The entry rate E_i into type-i parking is the integral of entry rates of different searcher types. The entry rate of type $\{c,t\}$ into type-i parking is the product of the following multipliers:

- the rate of searcher entry into the model, $a(c,t)$;
- the expected duration of search, $\frac{1}{r(c,t)}$;
- expected number of type-i bays sampled per unit of time, $\frac{s_i(c,t)}{S_i}$;
- probability of taking a vacant type-i bay if sampled, $p_i(c,t)$.

Hence, $E_i = \int_{c,t} \frac{a(c,t)s_i(c,t)p_i(c,t)}{r(c,t)S_i}\mathrm{d}c\mathrm{d}t$.

The rate of exit from type-i parking X_i is the inverse of the expected duration of parking. Hence,

$$X_i = \frac{E_i}{\int_{c,t} \frac{a(c,t)s_i(c,t)p_i(c,t)t}{r(c,t)S_i}\mathrm{d}c\mathrm{d}t},$$

and therefore

$$T_i = \int_{c,t} \frac{a(c,t)s_i(c,t)p_i(c,t)t}{r(c,t)S_i} \mathrm{d}c\mathrm{d}t. \tag{5}$$

We will nickname T_i as the *demand for parking* of type i.

3. Socially optimal search

3.1. Planner's objective

In this section, we investigate how to maximize social welfare using the steering strategy $s_i(c,t)$ and parking strategy $p_i(c,t)$ as controls. Because the inflow of searchers is exogenous, maximization of social welfare amounts to minimization of social search costs induced by all motorists searching at a representative moment of time. The social search cost induced by motorists of type $\{c,t\}$ is the product of

- the rate of searcher entry into the model, $a(c,t)$;
- the expected duration of search, $\frac{1}{r(c,t)}$;
- the social search cost per unit of time, c.

Hence, the social planner's objective is to minimize

$$G \equiv \int_{c,t} \frac{a(c,t)c}{r(c,t)} \mathrm{d}c\mathrm{d}t, \tag{6}$$

subject to constraints (3,4,5). The corresponding Lagrangian can be written as follows:

$$\begin{aligned} \mathcal{L}(s_i(c,t), p_i(c,t), r(c,t), T_i) = \int_{c,t} \frac{a(c,t)c}{r(c,t)} \mathrm{d}c\mathrm{d}t + \int_{c,t} \left[r(c,t) - \sum_{i=\{1,2\}} \frac{s_i(c,t)p_i(c,t)}{1+T_i} \right] \mu(c,t) \mathrm{d}c\mathrm{d}t \\ + \sum_{i=\{1,2\}} \left[\int_{c,t} \frac{a(c,t)ts_i(c,t)p_i(c,t)}{r(c,t)S_i} \mathrm{d}c\mathrm{d}t - T_i \right] \lambda_i \to \min. \end{aligned} \tag{7}$$

Prior to formal derivation of optimal search strategy, we point out the following useful observation.

Proposition 1. *For any given searcher type $\{c, t\}$, optimal parking strategy for at least one type of parking is maximized:* $\max\{p_1(c,t), p_2(c,t)\} = 1$.

In simple words, for any searcher, there exists a type of parking there she will certainly park once a vacancy is found.

Proof. Suppose for some $\{c, t\}$ both $p_1(c,t) < 1$ and $p_2(c,t) < 1$. According to the system (3,4,5), a proportionate increase in $p_1(c,t)$ and $p_2(c,t)$ will increase $r(c,t)$ by the same proportion, while keeping T_i and q_i unchanged and thus having no effect on expected outcomes of other searchers. But that makes the objective function (6) smaller, improving social welfare. ■

3.2. First-order conditions

We now characterize the first-order conditions of the social optimum. First, for a given searcher type $\{c, t\}$, we minimize (7) with respect to the steering strategy $s_1(c,t)$, accounting for the fact that $\mathrm{d}s_2(c,t) \equiv -\mathrm{d}s_1(c,t)$:

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}s_1(c,t)} = \left[-\frac{p_1(c,t)}{1+T_1} + \frac{p_2(c,t)}{1+T_2}\right]\mu(c,t) + \frac{a(c,t)t}{r(c,t)}\left[\frac{\lambda_1 p_1(c,t)}{S_1} - \frac{\lambda_2 p_2(c,t)}{S_2}\right] \begin{array}{l} \geq 0, s_1(c,t) = \underline{s} \\ = 0, s_1(c,t) \in (\underline{s}, \bar{s}) \\ \leq 0, s_1(c,t) = \bar{s} \end{array} . \quad (8)$$

The first-order condition for optimal parking strategy is as follows:

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}p_i(c,t)} = -\frac{s_i(c,t)}{1+T_i}\mu(c,t) + \frac{a(c,t)t}{r(c,t)}\frac{s_i(c,t)}{S_i}\lambda_i \begin{array}{l} \geq 0, p_i(c,t) = 0 \\ = 0, p_i(c,t) \in (0,1) \\ \leq 0, p_i(c,t) = 1 \end{array} . \quad (9)$$

Optimal chance of success $r(c,t)$ must be inside the unit interval, hence we have

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}r(c,t)} = -\frac{a(c,t)c}{r^2(c,t)} + \mu(c,t) - \frac{a(c,t)t}{r^2(c,t)} \sum_i \frac{s_i(c,t)p_i(c,t)\lambda_i}{S_i} = 0,$$

from which we can derive the shadow value of the chance of success:

$$\mu(c,t) = \frac{a(c,t)}{r^2(c,t)} \left[c + t \sum_i \frac{s_i(c,t)p_i(c,t)\lambda_i}{S_i} \right]. \tag{10}$$

Finally, from the first-order condition for optimal demand T_i, which is optimally positive and finite, we can derive the shadow value of such demand:

$$\lambda_i = \int_{c,t} \frac{s_i(c,t)p_i(c,t)\mu(c,t)}{(1+T_i)^2} \mathrm{d}c\mathrm{d}t. \tag{11}$$

3.3. Optimal parking strategy

Combining (9) with (10) yields the following optimality condition for the parking strategy:

$$\frac{1}{r(c,t)} \left[\frac{c}{t} + \sum_j \frac{s_j(c,t)p_j(c,t)\lambda_j}{S_j} \right] - \frac{(1+T_i)\lambda_i}{S_i} \begin{array}{l} \leq 0, p_i(c,t) = 0 \\ = 0, p_i(c,t) \in (0,1) \\ \geq 0, p_i(c,t) = 1 \end{array} . \tag{12}$$

Note that the term in square brackets in (12) is the same for both types of parking. Because labeling of parking types was arbitrary, suppose without loss of generality that

$$\frac{\lambda_1(1+T_1)}{S_1} \leq \frac{\lambda_2(1+T_2)}{S_2}. \tag{13}$$

Then, (12) implies that

$$p_1(c,t) \geq p_2(c,t), \forall c,t. \tag{14}$$

But then, proposition 1 also implies that $p_1(c,t) = 1$ for all searcher types $\{c,t\}$. Therefore, we can conclude that parking privileges, if they exist, should cover only part of all parking (type-2 in our notation); the remaining parking (type-1) should be available to any searcher who finds vacancy there.

To find optimal strategy for type-2 parking, rewrite (12) for $i = 2$, substituting $p_1(c,t) = 1$, as follows:

$$\frac{c}{t} + \frac{s_1(c,t)\lambda_1}{S_1} + \frac{s_2(c,t)p_2(c,t)\lambda_2}{S_2} - r(c,t)\frac{\lambda_2(1+T_2)}{S_2} \begin{array}{l} \leq 0, p_2(c,t) = 0 \\ = 0, p_2(c,t) \in (0,1) \\ \geq 0, p_2(c,t) = 1 \end{array},$$

which can be rewritten, recalling (3) and (4), as

$$\frac{c}{t} - \frac{s_1(c,t)}{1+T_1}\left[\frac{\lambda_2(1+T_2)}{S_2} - \frac{\lambda_1(1+T_1)}{S_1}\right] \begin{array}{l} \leq 0, p_2(c,t) = 0 \\ = 0, p_2(c,t) \in (0,1) \\ \geq 0, p_2(c,t) = 1 \end{array}. \tag{15}$$

Denote

$$L_0 \equiv \frac{\lambda_2(1+T_2)}{S_2} - \frac{\lambda_1(1+T_1)}{S_1} \tag{16}$$

and $k_1 \equiv \frac{s_1(c,t)}{1+T_1}L_0$; both are non-negative by assumption (13). But then, we conclude that a searcher of type $\{c,t\}$ should *never* use type-2 parking, $p_2(c,t) = 0$, if their cost-to-duration ratio $\frac{c}{t}$ is below k_1; a searcher should *always* use vacant type-2 parking they find, $p_2(c,t) = 1$, if such ratio is above k_1. Those with $\frac{c}{t} = k_1$ are indifferent between various values of $p_2(c,t)$, but assumption 1 implies that the population share of such searchers is zero hence they can be ignored.

3.4. No-steering case

Consider a special case when no steering is possible, i.e. $s_1(c,t)$ can take only one possible value S_1, meaning that draws of parking bays are completely random. In such environment, a searcher $\{c, t\}$ who uses the *fast parking strategy*, i.e. always takes the first available bay $p_i = 1, i \in \{1, 2\}$, has the chance of success equal to $1 - D$, which is also equal to the fraction of unoccupied bays in steady state (cf.(1)). By assumptions of the model, such probability does not depend on parking strategy of other searchers. A searcher $\{c, t\}$ pursuing any other parking strategy, i.e. $p_2(c, t) < 1$, will have a smaller chance of success and will therefore search longer. Thus, the aggregate search cost is minimized when everyone uses the fast strategy of search, meaning that any parking privilege cannot be optimal.

3.5. Optimal steering strategy

The rest of the paper except section 7 assumes that searchers can indeed steer their search, i.e. $\underline{s} < \bar{s}$. In the current section, given optimal parking strategy $p_i(c, t)$, we calculate the optimal steering strategy $s_i(c, t)$. The searchers that are excluded from type-2 parking, i.e. those with $c < k_1 t$ and using the parking strategy $\{p_1(c, t) = 1, p_2(c, t) = 0\}$, have the following chance of success (cf.(3,4)): $r(c, t) = \frac{s_1(c,t)}{1+T_1}$. A comparison of the latter equation with the definition of T_1 in (5) implies that an increase in $s_1(c, t)$ increases $r(c, t)$ by the same proportion without changing T_1, which reduces the social search cost (6). Therefore, for such searchers the optimal steering strategy is to maximize the chance of sampling type-1 parking, $s_1(c, t) = \bar{s}$. Which is very intuitive given the fact that they can use only this type of parking.

For the searchers with $c > k_1 t$ who use the fast parking strategy, optimal steering strategy is less straightforward. For them, (8) can be rewritten as

$$
\begin{aligned}
&\left[-\frac{1}{1+T_1}+\frac{1}{1+T_2}\right]\mu(c,t)+\frac{a(c,t)t}{r(c,t)}\left[\frac{\lambda_1}{S_1}-\frac{\lambda_2}{S_2}\right] \\
&\underbrace{=}_{\text{cf.(10)}}\frac{a(c,t)}{r^2(c,t)}\left[\left(-\frac{1}{1+T_1}+\frac{1}{1+T_2}\right)\left(c+t\sum_i\frac{s_i(c,t)\lambda_i}{S_i}\right)+t\left(\sum_i\frac{s_i(c,t)}{1+T_i}\right)\left(\frac{\lambda_1}{S_1}-\frac{\lambda_2}{S_2}\right)\right] \\
&=\frac{a(c,t)}{r^2(c,t)(1+T_1)(1+T_2)}\left[(T_1-T_2)c+t\left(\frac{\lambda_1(1+T_1)}{S_1}-\frac{\lambda_2(1+T_2)}{S_2}\right)\right]\begin{array}{l}\geq 0, s_1(c,t)=\underline{s}\\ =0, s_1(c,t)\in(\underline{s},\bar{s})\\ \leq 0, s_1(c,t)=\bar{s}\end{array}.
\end{aligned}
\tag{17}
$$

We now analyze two potential equilibrium patterns.

3.5.1. No privileged parking

Proposition 2. *If (13) is equality, then $T_1 = T_2$.*

Proof. If (13) is equality, then according to (15) everyone should follow the fast parking strategy $p_i(c,t) = 1, \forall i, c, t$. Furthermore, the sign of (17) is equal to the sign of $T_1 - T_2$. Suppose $T_1 < T_2$; then all searchers steer towards type-1 parking, $s_1(c,t) = \bar{s}, \forall c, t$. The assumptions of section 2.3 then imply that

$$\frac{s_1(c,t)}{S_1} \geq 1 \geq \frac{s_2(c,t)}{S_2}, \forall c, t. \tag{18}$$

But then, from the definition (5) of T_i, it follows that $T_1 \geq T_2$, contradicting the initial assumption. Similar logic excludes $T_1 > T_2$. ■

In simple words, (13) being equality means that the social cost of occupying both types of parking is the same, which further means that all searchers should use the fast parking strategy, and that they can choose an arbitrary steering strategy $s_i(c,t) \in [\underline{s}, \bar{s}]$ as long as the aggregate proportion of type-i parking being drawn equals the share of such parking in aggregate supply.

3.5.2. Privileged parking

Proposition 3. *If (13) is strict inequality, then $T_1 > T_2$.*

Proof. Suppose (13) is strict inequality but $T_1 \leq T_2$. Then (17) is strictly negative, meaning that all searchers steer towards type-1 parking, $s_1(c,t) = \bar{s}, \forall c, t$, thus (18) is true. This inequality together with (14) implies that $T_1 \geq T_2$ according to (5). This excludes $T_1 < T_2$.

Suppose further that $T_1 = T_2$. From (11), we can rewrite the terms in (13) as

$$\frac{\lambda_i(1+T_i)}{S_i} = \int_{c,t} \frac{s_i(c,t)}{S_i} \frac{p_i(c,t)\mu(c,t)}{1+T_i} \mathrm{d}c\mathrm{d}t.$$

But then, (14) and (18) imply that $\frac{\lambda_1(1+T_1)}{S_1} \geq \frac{\lambda_2(1+T_2)}{S_2}$, contradicting the assumption of this proposition. ■

Given $T_1 > T_2$, the optimal steering strategy can be formulated from (17) as follows:

$$s_1(c,t) \begin{array}{ll} = \bar{s}, & \frac{c}{t} < k_2 \\ \in [\underline{s}, \bar{s}], & \frac{c}{t} = k_2 \\ = \underline{s}, & \frac{c}{t} > k_2 \end{array}, \tag{19}$$

where $k_2 \equiv \frac{1}{T_1 - T_2} L_0$. It is trivial to see that $k_2 > k_1$, hence searchers with $\frac{c}{t} = k_1$ steer towards type-1 parking and hence k_1 can be rewritten as $k_1 = \frac{\bar{s}}{1+T_1} L_0$.

3.6. Summary of optimal strategy

As a summary of the above discussion, all searchers can be divided into three groups.

3.6.1. Searchers excluded from privileged parking

Searchers with lowest cost-to-duration ratio, $\frac{c}{t} < k_1$, can use only type-1 parking and should naturally steer towards that parking. We will label these searchers as "group x". The mass of such motorists parked at any time is $D_x(k_1) \equiv \int_t \int_{c=0}^{k_1 t} a(c,t) t \mathrm{d}c\mathrm{d}t$; the flow of social costs caused by searching group-x motorists is $G_x(k_1) \equiv \int_t \int_{c=0}^{k_1 t} a(c,t) c \mathrm{d}c\mathrm{d}t$.

In the no-privileged-parking scenario, we have that $k_1 = 0$ hence group x does not exist, i.e. $D_x(0) = 0$ and $G_x(0) = 0$.

3.6.2. Searchers prioritizing non-privileged parking

Those with intermediate cost-to-duration ratio, $k_1 < \frac{c}{t} < k_2$, use the fast parking strategy while steering towards type-1 (more occupied) parking. We will denote them as group-y searchers. The mass of such motorists parked in steady state is $D_y(k_1, k_2) \equiv \int_t \int_{c=k_1 t}^{k_2 t} a(c,t) t \mathrm{d}c\mathrm{d}t$; the search by these motorists causes the flow of social cost equal to $G_y(k_1, k_2) \equiv \int_t \int_{c=k_1 t}^{k_2 t} a(c,t) c \mathrm{d}c\mathrm{d}t$.

In equilibrium without privileged parking, all searchers are indifferent between various steering strategies; we can assume that those with $\frac{c}{t} < \hat{k}_2$ behave as group-y searchers, i.e. choose $s_1(c,t) = \bar{s}$. The specific value of $\hat{k}_2$ is defined below in (20). The mass of parked motorists of this type is $D_y(0, \hat{k}_2)$ while the flow of social cost is $G_y(0, \hat{k}_2)$.

3.6.3. Privileged searchers

Finally, searchers with highest cost-to-duration ratio, $\frac{c}{t} > k_2$, should use the fast parking strategy and steer towards type-2 (privileged) parking, so that $s_1(c,t) = \underline{s}$. Denote them as "group z". The mass of parked motorists of this type is $D_z(k_2) \equiv \int_t \int_{c=k_2 t}^{\infty} a(c,t) t \mathrm{d}c\mathrm{d}t$ while the flow of social cost is $G_z(k_2) \equiv \int_t \int_{c=k_2 t}^{\infty} a(c,t) c \mathrm{d}c\mathrm{d}t$.

By construction, we have that $D_x(k_1) + D_y(k_1, k_2) + D_z(k_2) \equiv D$.

In the scenario without parking privileges, we can assume that those with $\frac{c}{t} > \hat{k}_2$ behave as group-z searchers. Because in such scenario group-x does not exist, we have that $D_y(0, \hat{k}_2) + D_z(\hat{k}_2) = D$. The requirement specified in section 3.5.1, that the share of type-1 parking being drawn by searchers equals its share in supply, can then be written as

$$D_y(0,\hat{k}_2)\bar{s} + D_z(\hat{k}_2)\underline{s} = (D - D_z(\hat{k}_2))\bar{s} + D_z(\hat{k}_2)\underline{s} = DS_1.$$

Hence, without parking privileges, we have

$$D_z(\hat{k}_2) = D\frac{\bar{s} - S_1}{\bar{s} - \underline{s}}, \tag{20}$$

which pins down the unique value of $\hat{k}_2$.

3.7. Ruling out non-discriminatory policy

Section 3.4 finds that non-discriminatory policy with equal occupancies $q_1 = q_2$ and all searchers using the fast parking strategy is optimal when there is no steering, $\underline{s} = \bar{s}$. Section 3.5 has found that such policy also meets all first-order conditions of social optimum when motorists can steer their search, $\underline{s} < \bar{s}$. But is it indeed optimal to rule out parking privileges in the latter scenario? This section proves that the answer is negative: despite the first-order conditions of optimality being met, non-discriminatory policy is inferior to optimal privileges when motorists can steer their search.

The method of proof is to construct a small deviation from a no-privilege equilibrium and demonstrate an improvement (decrease) in the social objective function (6). Without privileges, both types of parking are equally occupied so that $q_i = D, \forall i$. Suppose we increase k_2 from $\hat{k}_2$ determined by (20) by a small amount, so that some searchers migrate from group z to group y and thus type-1 occupancy increases by an infinitesimal amount δ: $q_1 = D + \delta$. According to (2), the type-2 (privileged) occupancy is then reduced to $q_2 = D - \frac{S_1}{S_2}\delta$.

Lemma 1. *The derivative* $\frac{\mathrm{d}\delta}{\mathrm{d}k_2}$ *at point* $\{\delta = 0, k_2 = \hat{k}_2\}$ *is positive and finite.*

The proof is in Appendix A.

Proposition 4. *A small increase of cutoff* k_2 *from initial value* $\hat{k}_2$ *reduces the social cost of search (6).*

Proof. The chance of success of searchers from group y is equal to $r_y(\delta) = \bar{s}(1-q_1) + (1-\bar{s})(1-q_2) = 1 - D - \frac{\bar{s}-S_1}{S_2}\delta$. Likewise, the chance of success of searchers from group z is $r_z(\delta) = \underline{s}(1-q_1) + (1-\underline{s})(1-q_2) = 1 - D + \frac{S_1-\underline{s}}{S_2}\delta$. The social cost of search is as follows (cf.(6)):

$$G(k_2) = \frac{G_y(0,k_2)}{r_y(\delta(k_2))} + \frac{G_z(k_2)}{r_z(\delta(k_2))}. \tag{21}$$

By differentiating (21) with respect to k_2 at point $\hat{k}_2$, recalling $\frac{\mathrm{d}G_y(0,k_2)}{\mathrm{d}k_2} = -\frac{\mathrm{d}G_z(k_2)}{\mathrm{d}k_2}$, and dropping the second-order terms, we obtain the following:

$$\frac{\mathrm{d}G(\hat{k}_2)}{\mathrm{d}k_2} = \frac{G_y(0,\hat{k}_2)}{(1-D)^2}\frac{\bar{s}-S_1}{S_2} - \frac{G_z(\hat{k}_2)}{(1-D)^2}\frac{S_1-\underline{s}}{S_2}. \tag{22}$$

From (20) and $D_y(0,\hat{k}_2) + D_z(\hat{k}_2) = D$, we have that $D_y(0,\hat{k}_2)(\bar{s}-S_1) = D_z(\hat{k}_2)(S_1-\underline{s})$. By dividing (22) by the latter equality we obtain that the sign of (22) is the same as the sign of $\frac{G_y(0,\hat{k}_2)}{D_y(0,\hat{k}_2)} - \frac{G_z(\hat{k}_2)}{D_z(\hat{k}_2)}$. Recalling the definitions of G_w and $D_w, w \in \{y,z\}$ in section 3.6, the ratio $\frac{G_w}{D_w}$ can be interpreted as the ratio of average cost c for group w to the average parking duration t for the same group. But because, for any given t, those in group z have greater $\frac{c}{t}$ ratios by definition of the group, we have that the $\frac{G_w}{D_w}$ ratio is greater for group z and thus (22) is negative. Which means that a small increase in k_2, that redistributes some searchers from group z to group y, decreases the social cost of search and thereby improves social welfare. ■

Once an increase in steering strategy cutoff k_2 is implemented, occupancies in the two types of parking become unequal, so that $q_1 > q_2$ and hence (cf.(4)) $T_1 > T_2$. But then, analysis of section 3.5 implies that (13) should be held with strict inequality, which further means that $k_1 > 0$ and searchers with $\frac{c}{t} < k_1$ should optimally be excluded from type-2 parking.

To conclude, if motorists can steer their search, no-privilege policy is not optimal despite the fact that all first-order conditions of optimality are met. Thus, the social optimum must feature unequal occupancies $q_1 > q_2$ and some searchers must be excluded from type-2 parking.

3.8. Low demand

How should optimal policy respond to periods of low demand, e.g. at night or on weekend? To fix ideas, assume that the inflow of searchers $a(c,t)$ proportionately decreases to zero for every $\{c,t\}$. Then, because finding vacancy is easier, the chance of success $r(c,t)$ rises to an upper bound, which means that the quantity $\mu(c,t)$ in (10) decreases at the same rate as $a(c,t)$. But then, λ_i in (11), L_0 in (16), and k_1 also decrease to zero at the same rate. Thus, we can conclude that during periods of near-zero demand, privileges are not optimal, and no searcher should be excluded from any parking bay.

4. Implementing optimal policy

This section studies how the government can implement the socially optimal policy characterized in section 3. Because a motorist's duration of parking t is known to the government while the cost of search c is not, regulation of parking can depend only on t. Furthermore, because entry $a(c,t)$ of each particular type of searchers is exogenous by a model assumption and does not respond to regulation, we will focus on the *difference* in government regulation across the two parking types. We will assume that, for a searcher with parking duration t, regulation consists of a monetary *premium* $L(t)$ for using type-2, rather than type-1, parking. Outcomes of many other regulations can be achieved by selecting an appropriate value of $L(t)$. For example a direct ban to use type-2 parking for t units of time is equivalent to $L(t) = \infty$.

As mentioned in the introduction, the cost of search c consists of (i) private cost c_1 and (ii) external cost c_0. The former category includes passengers' value of time and the cost of operating a vehicle; the latter includes cruising-for-parking externalities, pollution, and accidents. We will assume that c_1 is heterogenous and is private information, while c_0 is common to all searchers and can be estimated by the government.

We will further assume that motorists only account for their private search cost c_1 when calculating their optimal search strategy.

A motorist with parameters $\{c_1, t\}$ selects her steering strategy $s_i(c_0 + c_1, t)$ and parking strategy $p_i(c_0 + c_1, t), \forall i$ to minimize her total cost of parking $C(c_1, t)$ which consists of the following elements:

- the private cost of drawing another parking bay;
- the chance of ending up in type-2 bay, times the premium paid for parking there;
- the chance of having to continue search, times the total cost of parking:

$$
\begin{aligned}
C(c_1,t) = \min_{s_i(\cdot),p_i(\cdot)} c_1 + \frac{s_2(c_0+c_1,t)p_2(c_0+c_1,t)}{1+T_2} L(t) + \left(1 - \sum_i \frac{s_i(c_0+c_1,t)p_i(c_0+c_1,t)}{1+T_i}\right) C(c_1,t) \\
= \min_{s_i(\cdot),p_i(\cdot)} \frac{c_1 + \frac{s_2(c_0+c_1,t)p_2(c_0+c_1,t)}{1+T_2} L(t)}{\sum_i \frac{s_i(c_0+c_1,t)p_i(c_0+c_1,t)}{1+T_i}}.
\end{aligned}
$$

By our earlier assumption, type-2 parking is less occupied (without loss of generality) which can be achieved by making it more expensive: $L(t) \geq 0$. Then, optimization of the above objective yields $p_1(c_0 + c_1, t) = 1$ for all searchers. Equilibrium type-2 parking strategy is

$$
p_2(c_0+c_1,t) \begin{array}{ll} = 1, & c_1 > \\ \in [0,1], & c_1 = \\ = 0, & c_1 < \end{array} \frac{L(t)\bar{s}}{1+T_1}, \tag{23}
$$

while the equilibrium steering strategy is

$$s_1(c_0+c_1,t) \begin{cases} = \bar{s}, & c_1 < \\ \in [\underline{s},\bar{s}], & c_1 = \dfrac{L(t)}{T_1-T_2}. \\ = \underline{s}, & c_1 > \end{cases} \tag{24}$$

4.1. No external search costs

When $c_0 = 0$, i.e. there are no search externalities, equilibrium search strategy is socially optimal iff $L(t) = tL_0$.

4.2. Positive external search costs

When $c_0 > 0$, we essentially have a positive lower bound on the social search cost $c = c_0 + c_1$ which, by assumption, is common to all searcher. But then, all searchers with a sufficiently short duration of parking belong to group z, i.e. they should be allowed to park anywhere and should target type-2 (privileged) parking. To achieve this in equilibrium, the government should not practice any parking discrimination for those parking short-term, i.e. $L(t) = 0$ for small t. In particular, *a vehicle that is loading or unloading passengers or cargo should be allowed to use all types of parking*; the cost of such parking should be the same for a given (short) parking duration at all types of parking.

For longer parking durations with $c_0 > 0$, derivation of optimal $L(t)$ is tedious and lacking an economic intuition; for this reason, we characterize only the range of possible values of $L(t)$ which is relatively easy to derive.

The upper bound $\hat{L}(t)$ on $L(t)$ is such that the boundary between group-y and group-z searchers is socially optimal. Consider a searcher who is optimally indifferent between groups y and z; according to (cf.(19), such searcher has $c_1 = \frac{L_0 t}{T_1-T_2} - c_0$ and parking duration t exceeding $\frac{c_0(T_1-T_2)}{L_0}$. From (24), we also have that $c_1 = \frac{\hat{L}(t)}{T_1-T_2}$ and hence $\hat{L}(t) = L_0 t - c_0(T_1 - T_2)$ which is non-negative for all relevant parking durations. Given such premium for type-2

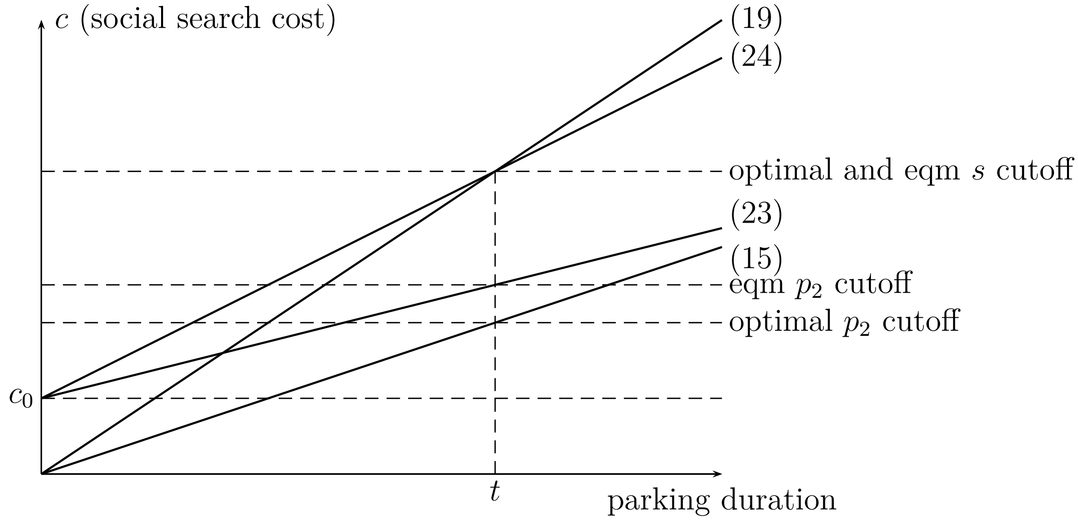

Figure 1: Optimal and equilibrium strategy cutoffs when premium is $\hat{L}(t)$

parking, according to (23), searchers are indifferent between being group-x and group-y (i.e. between skipping and taking a vacant type-2 parking) iff $c_1 = \frac{\hat{L}(t)\bar{s}}{1+T_1} = \frac{L_0 t\bar{s}}{1+T_1} - \frac{c_0(T_1-T_2)\bar{s}}{1+T_1}$, which exceeds the socially optimal value of (cf.(15)) $c_1 = \frac{L_0 t\bar{s}}{1+T_1} - c_0$. Figure 1 illustrates the cutoff values of c, for given t, in the social optimum and under the premium $\hat{L}(t)$.

By analogy, the lower bound $\underline{L}(t)$ on $L(t)$ makes the boundary between group-x and group-y searchers socially optimal. From (15), such boundary exists for $t \geq \frac{c_0(1+T_1)}{\bar{s}L_0}$ and is characterized by $c_1 = \frac{L_0 t\bar{s}}{1+T_1} - c_0$. By comparing the latter against (23), we have that $\underline{L}(t) = L_0 t - \frac{c_0\bar{s}}{1+T_1} < \hat{L}(t)$. If the premium of $\underline{L}(t)$ is used, the boundary between groups y and z characterized by (24) is below the socially optimal threshold (19).

The range of potentially optimal premia $L(t)$ is illustrated on figure 2.

4.3. Homogenous search cost

In a special case when all searchers have the same cost of search c_1, optimal search strategy cutoffs for parking duration are determined as follows. The duration cutoff t_p between $p_2 = 0$ (i.e. group x) and $p_2 = 1$ (group y) is found from $\frac{c_0+c_1}{t_p} = \frac{\bar{s}L_0}{1+T_1}$; the cutoff t_s between $s_1 = \bar{s}$ (group y) and $s_1 = \underline{s}$ (group z) is determined by $\frac{c_0+c_1}{t_s} = \frac{L_0}{T_1-T_2}$. It turns out that the same strategy cutoffs can be achieved by a parking premium regulation, by

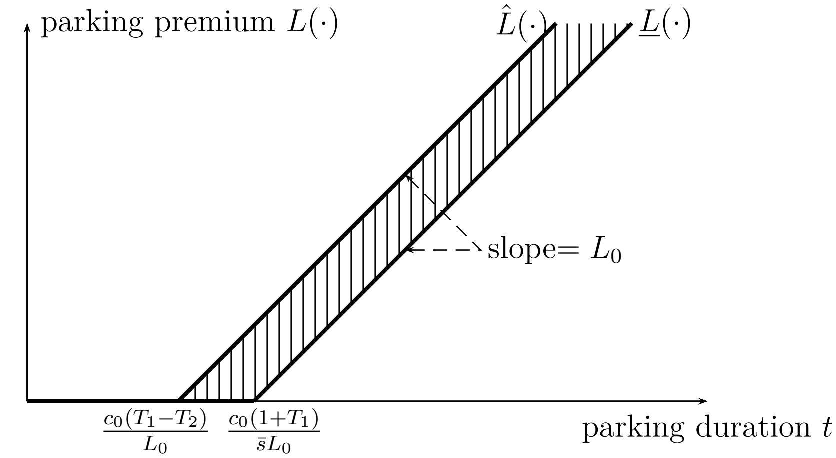

Figure 2: The bounds on optimal type-2 parking premium

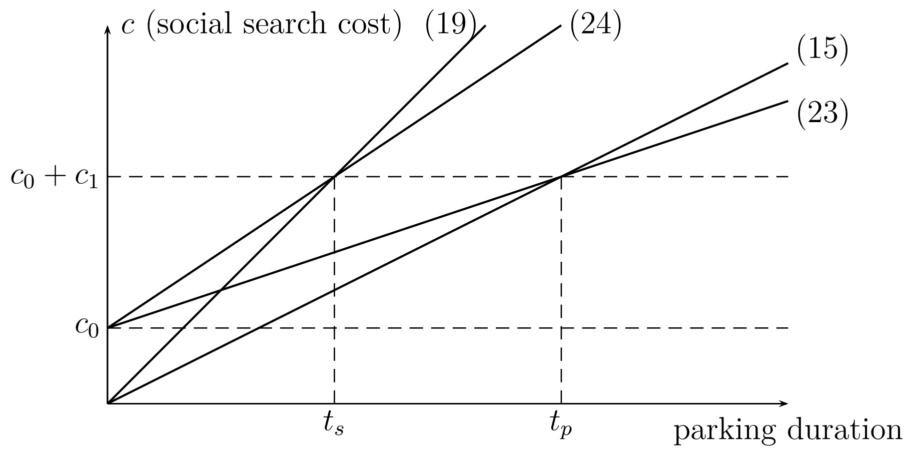

Figure 3: Optimal and equilibrium strategy cutoffs under homogenous search cost

making $L(t) = \frac{c_1}{c_0+c_1} L_0 t$. It is then straightforward to verify that (23) holds with equality iff $t = t_p$ while (24) is equality iff $t = t_s$. Hence, parking premium regulation allows to achieve social optimum if searchers vary only in desired parking duration. Figure 3 illustrates such equilibrium.

5. Empirical relevance

Suppose a government decides to adopt recommendations of this paper, divide municipal parking supply in a certain area into two types, and introduces a premium $L(t)$ for using type-2 parking for t units of time. How to assess whether this parking premium is optimal? This section describes how to use administrative data on parking demand under given $L(\cdot)$

for such assessment.

To make such judgements, all model parameters should be calibrated or estimated. To assess the externality of a cruising vehicle c_0, traffic flow theory can be applied. For example one can use "bathtub" models (Arnott (2013), Fosgerau (2015)) which relate average travel speed to the density of vehicles on the road in the area of interest. As those searching for parking add to the density, they cause delays of other vehicles. Arnott et al. (2015) incorporate these effects into their model of search for parking. By calibrating parameters of a bathtub model, one can assess the value of c_0 under prevailing traffic conditions.

The steering parameters $\bar{s}, \underline{s}$ depend on the street pattern and location of the different types of parking. For example to assess $\bar{s}$, one should study what is the maximal share of type-1 parking that a searching vehicle can encounter while cruising within a designated search zone. Section 6 provides an example of calibration of $\bar{s}$ and $\underline{s}$.

Occupancy levels q_i and therefore demand T_i can be observed directly from the data on use of both parking types. Then, equilibrium choice of parking $p_i(c,t)$ and steering $s_i(c,t)$ strategies can be assessed from (23) and (24), respectively. Given all above information, the chance of success (3) can be calculated.

To find the remaining unknowns, the distribution of searchers by type $a(c,t)$ is needed. While the distribution of parking durations can be estimated from data on parking demand, the cost of search c is private information and its distribution can be recovered only partly, through the fact that those with higher c are more likely to end up in type-2 parking. To address the problem, $a(c,t)$ can be parameterized as follows: $a(c,t) \equiv b(t)g(c,\alpha(t))$. Here $b(t)$ is the inflow, per unit of time, of all searchers willing to park for t units of time, so that $\int_t b(t)\mathrm{d}t = D$. Function $b(\cdot)$ can be estimated from administrative data on parking demand. The function $g(c,\alpha(t))$ is the probability distribution function of search costs for those willing to park for t units of time, with lower bound c_0 and mean $c_0 + \alpha(t)$. The shape of the function cannot be derived from the data and should be assumed, e.g. exponential

$g(c,\alpha) = \frac{1}{\alpha}\exp\left(-\frac{c-c_0}{\alpha}\right), c \geq c_0$. The predicted share of searchers with duration t who end up in privileged parking increases with mean search cost $\alpha(t)$, hence the latter can be estimated by matching such predicted share to empirically observed share, for every duration t.

The shadow costs $\mu(c,t)$ and λ_i can be found by solving the system (10,11), using the above described estimates of all other ingredients.

With all model parameters estimated, one can calculate the value of L_0 from (16) and then make judgements whether or not the existing policy $L(\cdot)$ is close to the optimal one formulated in section 4. If not, comparison of existing and optimal policies can also suggest in what direction the existing policy $L(\cdot)$ should be changed.

6. Calibrated example

Consider the following stylized example. Suppose a city center consists of North-South "avenues" and East-West "streets", all of which are one-way. Suppose further that the streets are three times more proximate to each other than the avenues (think of Manhattan grid) and thus have three times more curbside. Also suppose that, because the avenues are more busy, the government decides to make parking there privileged. Hence, street curbs offer type-1 parking such that $S_1 = 0.75$, while the avenue curbs have type-2 parking.

Because both streets and avenues are one-way, a searcher who prioritizes type-1 (street) parking is best-off using a snake search pattern, gradually drifting south: start at the northernmost street in the designated search area, go west for two blocks, use an avenue to travel one block south, go back east for two blocks, use another avenue to travel one more block south, turn west again, etc. Given such search pattern, 6 out of 7 parking bays encountered by the searcher will be type-1, hence $\bar{s} = \frac{6}{7}$.

A motorist prioritizing type-2 parking can use a similar pattern, starting at the easternmost avenue of the search area and gradually drifting west. Because the streets are much more frequent, we will assume that a searcher travels for four blocks on a given avenue be-

Parameter		Mean occupancy 90%		Mean occupancy 98%	
Verbal description	Math description	Optimal privilege	No privilege	Optimal privilege	No privilege
Type-1 occupancy, %	q_1	92.30	90.0	98.52	98.00
Type-2 occupancy, %	q_2	83.11	90.0	96.43	98.00
Share of searchers excluded from type-2 parking, %	$\frac{\int_{\frac{c}{t}<k_1} a(c,t)\mathrm{d}c,t}{\int_{c,t} a(c,t)\mathrm{d}c,t}$	33.25	0.0	33.51	0.0
Group-x search duration, sec	$\frac{1}{r(c,t)}, \frac{c}{t} < k_1$	36.3	24.0	190	120
Group-y search duration, sec	$\frac{1}{r(c,t)}, k_1 \leq \frac{c}{t} < k_2$	26.6	24.0	135	120
Group-z search duration, sec	$\frac{1}{r(c,t)}, \frac{c}{t} \geq k_2$	18.5	24.0	90	120
Type-2 parking premium, $ per hour	L_0	0.30	0	1.59	0
Expected search cost per motorist, $	$E_{c,t}\left(\frac{c}{r(c,t)}\right)$	0.36	0.40	1.79	2.00

Table 1: Steady states under optimal and egalitarian parking regulations

fore turning west. Then, the proportion of street bays to avenue bays encountered by the searcher is 3:4, hence $\underline{s} = \frac{3}{7}$.

We will assume that demand parking duration t and search cost c are each exponentially distributed and independent from each other, such that the p.d.f. of the distribution is $\frac{1}{EtEc}\exp\left(-\frac{t}{Et}\right)\exp\left(-\frac{c}{Ec}\right)$. The mean duration of parking is assumed to be one hour; Zakharenko (2016) estimates that 1500 parking bays can be sampled during this time, hence we assume $Et = 1500$.[3] The mean cost of search, which also includes the time cost of walking between the parked automobile and final destination, is assumed to be $1 per minute of cruising; the cost is assumed to be entirely private.[4] As we assume that a searcher can inspect 25 bays per minute of cruising, we have that $Ec = 1/25$ per draw of a parking bay.

For comparison purposes, we calculate optimal policy for two distinct values of overall inflow of searchers and corresponding levels of congestion. In example 1, searcher inflow is such that the steady-state occupancy is 90%. In example 2, it is 98%. Table 1 reports the results.

[3] van Ommeren and McIvor (2018) come up with a similar estimate of 1600.

[4] Zakharenko (2016) uses a similar estimate. van Ommeren and McIvor (2018) use a much smaller estimate of 25 Australian dollars per hour, but they emphasize that they exclude the walking costs.

The two experiments with different mean occupancy levels produce comparable results, after adjusting for a five-fold change in the vacancy rate. In both cases, the vacancy rate in the privileged type-2 parking is double that of non-privileged type-1 parking. Both experiments conclude that only one-third of all searchers, those with lowest $\frac{c}{t}$ ratio, should be excluded from the privileged parking. Thus, we can conclude that currently practiced privileged parking policies which exclude the vast majority of searchers (e.g. loading zones, taxi stands, etc.) are excessively restrictive.

The experiments also find that optimal privilege policy reduces the social costs of search by about 10%, regardless of overall congestion levels.

7. Special needs parking

7.1. Problem description

Some motorists cannot use standard parking bays and require specially designed ones to park their vehicles. For example accessible parking designed for people with disabilities usually has larger dimensions, allowing users to unload their mobility devices. Additional features may include proximity of a ramp to get on a sidewalk, and removal of curbside storm drains for safety of the disabled. Another example of special needs parking is the one for electric vehicles. As these vehicles require lengthy charging periods, they typically combine charging with parking and therefore can only park at bays equipped with relevant infrastructure.

This section assumes that a given fraction of all motorists have such special needs, and seeks answers to the following questions. Assuming special needs parking is more costly, what share of all parking should be modified? Which of ordinary motorists, if any, should be allowed to use the special needs parking?

7.2. The model

To answer these questions, we modify the model of this paper as follows. The unitary parking supply is endogenously divided into type-1 *regular* parking with share S_1 and type-2 *special needs* parking with remaining share $S_2 = 1 - S_1$. As type-2 parking is technologically more sophisticated, each bay costs extra f dollars per unit of time to the society.

All searchers are also divided into *regular* and *special needs*. Regular searchers are able to use any parking bay. The inflow of regular motorists with search cost c and parking duration t is $a(c,t)$; the stock of such motorists parked in steady state is $D = \int_c \int_t a(c,t)t\mathrm{d}c\mathrm{d}t$. Special needs searchers, in turn, can use only type-2 parking. As their search strategy is trivial, their distribution across c and t is immaterial; we will only assume that their inflow per unit if time is A_n, mean parking duration t_n, and mean search cost c_n. Thus, the total mass of vehicles parked at any time is $D + A_n t_n$.

Contrary to section 3.4, this section finds that special needs parking can be privileged even if all motorists cannot steer their search. To make the paper shorter, we will make that simplifying assumption. As many governments aim to disperse accessible parking across all urban blocks, rather than concentrate them in some blocks, increasing or decreasing the share of such parking along any search route is indeed problematic. Given such exogenous search strategy $s_i(c,t) = S_i$, optimal policy consists of finding the optimal parking strategy $p_i(c,t)$ for regular searchers, and of the optimal share of each type of parking S_i. We now reiterate the steps of section 3 to find the optimal policy in the new environment.

The chance of success of a regular searcher is redefined from (3) as $r(c,t) = \sum_{i\in\{1,2\}} \frac{S_i p_i(c,t)}{1+T_i}$, while the chance of success of a special-needs searcher is $r_n = \frac{S_2}{1+T_2} = S_2(1-q_2)$. Demand for parking is redefined from (5) as $T_1 = \int_{c,t} \frac{a(c,t)p_1(c,t)t}{r(c,t)}\mathrm{d}c\mathrm{d}t$ and $T_2 = \int_{c,t} \frac{a(c,t)p_2(c,t)t}{r(c,t)}\mathrm{d}c\mathrm{d}t + \frac{A_n t_n}{r_n}$. The objective function to be minimized now includes the social cost of search, redefined from

(6), as well as the additional cost of special needs parking:

$$G = \int_{c,t} \frac{a(c,t)c}{r(c,t)} \mathrm{d}c\mathrm{d}t + \frac{A_n c_n}{r_n} + f S_2. \tag{25}$$

7.3. Optimal parking strategy

To find optimal parking strategy $p_i(c,t)$, we minimize the Lagrangian equal to (cf.(7))

$$\begin{aligned}\mathcal{L} = \int_{c,t} \frac{a(c,t)c}{r(c,t)} \mathrm{d}c\mathrm{d}t + \frac{A_n c_n}{r_n} + \int_{c,t} \left[r(c,t) - \sum_i \frac{S_i p_i(c,t)}{1+T_i} \right] \mu(c,t) + \left[r_n - \frac{S_2}{1+T_2} \right] \mu_n + \\ \lambda_1 \left[\int_{c,t} \frac{a(c,t)p_1(c,t)t}{r(c,t)} \mathrm{d}c\mathrm{d}t - T_1 \right] + \lambda_2 \left[\int_{c,t} \frac{a(c,t)p_2(c,t)t}{r(c,t)} \mathrm{d}c\mathrm{d}t + \frac{A_n t_n}{r_n} - T_2 \right], \end{aligned} \tag{26}$$

while keeping S_i constant. Proposition 1 remains valid in the environment of this section, hence we have $\max_i(p_i(c,t)) = 1, \forall c,t$.

Optimization of (26) yields the first-order condition of optimal parking strategy similar to (12):

$$\frac{1}{r(c,t)} \left[\frac{c}{t} + \sum_j p_j(c,t)\lambda_j \right] - \frac{(1+T_i)\lambda_i}{S_i} \begin{array}{l} \leq 0, p_i(c,t) = 0 \\ = 0, p_i(c,t) \in (0,1) \\ \geq 0, p_i(c,t) = 1 \end{array}, \tag{27}$$

hence the sign of $p_2(c,t) - p_1(c,t)$ is, again, inversely related to the sign of $L_0 = \frac{\lambda_2(1+T_2)}{S_2} - \frac{\lambda_1(1+T_1)}{S_1}$. In section 3, we could assume that $L_0 \geq 0$ without loss of generality, because the two parking types were ex-ante identical. This is no longer the case, so the sign of L_0 has to be proven.

Proposition 5. *Under socially optimal policy, we have that $L_0 \geq 0$.*

Proof. Suppose the inverse, that $L_0 < 0$. Then, from (27) combined with proposition 1, regular searchers with $\frac{c}{t} \leq \frac{S_1}{1+T_1}|L_0|$ will take only type-2 parking: $p_1(c,t) = 0, p_2(c,t) = 1$; denote them "group-$v$". The remaining regular searchers with use the fast parking strategy, $p_i(c,t) = 1, \forall i$, which makes them group-y in the language of section 3.6. Because the probability of drawing each parking type is exogenous, the chance of success of group-y searchers is exogenous too and is equal to population share of vacant bays,

$$r_y = 1 - D - A_n t_n. \tag{28}$$

Group-v searchers have a lower chance of success. Consider all group-v searchers adopting group-y strategy (i.e. increasing $p_1(c,t)$ to the maximum). The welfare of these searchers is increased as they find vacancy faster; the welfare of initial group-y searchers is unchanged as their chance of success does not depend on strategies of others. As some of group-v searchers now end up in type-1 parking, the welfare of special-needs searchers also increases, as their chance of success r_n rises due to type-2 parking becoming less congested. Thus, we have indicated a Pareto-improvement, which contradicts optimality of initial allocation. ■

Given $L_0 \geq 0$, those with $\frac{c}{t} < k_1 \equiv \frac{L_0 S_1}{1+T_1}$ will use type-1 parking only ($p_2(c,t) = 0$) so we label them "group-x". Their chance of success is

$$r_x = \frac{S_1}{1+T_1} = S_1(1-q_1). \tag{29}$$

Note that the chances of success of group x and of special needs parkers add up to exogenously given population-average vacancy $r_x + r_n = r_y$. This means that a rise in cutoff k_1, by increasing the share of group-x searchers and thus making type-1 parking more congested, will reduce r_x while increasing r_n by the same amount.

7.4. Solving for social optimum

Using the functions $G_x(k_1)$ and $G_y(k_1,\infty)$ defined in section 3.6, and accounting for the dependence of r_x and r_n on k_1 and S_1, we can rewrite the objective (25) as follows:

$$\min_{k_1,S_1} \frac{G_x(k_1)}{r_x(k_1,S_1)} + \frac{G_y(k_1,\infty)}{r_y} + \frac{A_n c_n}{r_n(k_1,S_1)} + f(1-S_1). \tag{30}$$

The equilibrium value of r_x can be found from (29) and from the fact that $T_1 = \frac{D_x(k_1)}{r_x} + \frac{D_y(k_1,\infty)}{r_y}$, as follows:

$$r_x(k_1, S_1) = \frac{r_y(S_1 - D_x(k_1))}{r_y + D_y(k_1, \infty)} = \frac{r_y(S_1 - D_x(k_1))}{1 - D_x(k_1) - A_n t_n}. \tag{31}$$

Note that, if $S_1 = 0$ (there is no type-1 parking), then $r_x = 0$ (those targeting type-1 parking will search forever), which also means $k_1 = 0$ (no one will pursue such strategy). We also have that the first term in (30) is zero (there are no group-x searchers hence no associated cost), while $r_n = r_y$ (the chance of success of special-needs parkers is maximized). Then, the first-order condition of optimality of $S_1 = 0$, accounting for $\mathrm{d}r_x = -\mathrm{d}r_n$, reads $\frac{\mathrm{d}G}{\mathrm{d}S_1} = \frac{A_n c_n}{r_y(1-A_n t_n)} - f \geq 0$.

The share of regular parking S_1 must also be less than unity so that there was room to accommodate special-needs searchers; the first-order condition for optimal $S_1 > 0$ then reads

$$\left(-\frac{G_x}{r_x^2} + \frac{A_n c_n}{r_n^2}\right) \frac{r_y}{1 - D_x(k_1) - A_n t_n} - f = 0. \tag{32}$$

The optimal chance of success r_x of group-x searchers must take an interior value, hence the first-order condition for optimal k_1 reads, referring to Appendix B for derivatives of G_w and $D_w, w \in \{x, y\}$,

$$k_1 H(k_1) \left(\frac{1}{r_x} - \frac{1}{r_y}\right) - \left(-\frac{G_x}{r_x^2} + \frac{A_n c_n}{r_n^2}\right) H(k_1) \frac{r_y(1 - S_1 - A_n t_n)}{(1 - D_x(k_1) - A_n t_n)^2} = 0. \tag{33}$$

Equations (32) and (33) determine the optimal values of threshold cost-duration ratio k_1 and share of type-1 parking S_1.

7.5. Exclusive policy is bad even for special-needs searchers

Currently, governments practice *exclusive policy* for special-needs parking, when all regular searchers are banned to use it. In the language of this paper, governments impose

$k_1 = \infty$. The previous section has proven that such policy does not maximize social welfare. The following proposition demonstrates that, if the shares of different parking types respond optimally to changes in k_1, admission of a small fraction of regular searchers to special-needs parking makes even special-needs searchers better off.

Proposition 6. *Decreasing k_1 from infinity to large finite number, while adjusting S_1 according to (32), increases the chance of success r_n of the special-needs searchers.*

Intuitively, admission of some regular searchers to special-needs parking increases occupancy of the latter, making special-needs searchers worse off. At the same time, such policy change makes the optimal share $S_2 = 1 - S_1$ of special-needs parking to increase, improving welfare of special-needs searchers. Proposition 6 states that, if policy changes are small, the positive welfare effect dominates the negative one.

Proof. At the initial policy $k_1 = \infty$, no regular searcher is admitted to type-2 parking, hence $D_y(k_1, \infty) = G_y(k_1, \infty) = 0$ while $D_x(k_1) = D$, hence (32) reads (cf.(28))

$$-\frac{G_x}{r_x^2} + \frac{A_n c_n}{r_n^2} = f. \tag{34}$$

According to Appendix B, we have that $\frac{\mathrm{d}G_x}{\mathrm{d}k_1} = k_1 \frac{\mathrm{d}D_x}{\mathrm{d}k_1}$. This means that, if k_1 is decreased from infinity to a large finite value so that $G_x(k_1)$ marginally changes, the change in $D_x(k_1)$ will be of smaller order of magnitude and thus can be ignored. Intuitively, when k_1 is large, then regular parkers who are admitted to type-2 parking have high cost of search c and low parking duration t, which means that their impact on occupancies is negligible relative to their impact on social cost of search. But that means that the optimality condition for S_1, (32), can still be approximated by (34) for large finite k_1. As the change in k_1 from infinity to a finite value decreases G_x, formula (34) combined with $\mathrm{d}r_n = -\mathrm{d}r_x$ implies that the chance of success r_n of special-needs searchers must increase. Intuitively, admission of high-value regular searchers to type-2 parking increases its social value, leading to increase in the optimal amount of such parking and thus improving r_n. ■

7.6. Numerical example

We illustrate the optimal special-needs parking policy by calibrating the model to a large city deciding how much accessible parking to supply and who should be allowed to use it.

The primary cost f of accessible parking is its increased size, which is necessary for safe exit of disabled passengers from the vehicle. We will assume that additional 10 square meters are needed for an accessible parking bay. Using the land value estimate of 16 million USD per acre (as in downtown Los Angeles), the extra cost of a type-2 parking bay is then approximately \$40000. Assuming the interest rate of 3.5%, the annuity equivalent of such cost is \$1400 per year, or \$26.92 per week. Assuming further that parking is only demanded 40 hours per week, we arrive at $f = 0.67$ per hour of parking.

For simplicity, the cost of search is assumed to be the same for all regular searchers and equal to \$1 per minute of cruising, as in section 6. The mean search cost of special-needs motorists is five times higher, \$5 per minute of cruising, primarily because cruising time is strongly correlated with the walking distance from vehicle to destination, and such walking is much more costly for the disabled. The mean duration of parking is 1 hour for both types of searchers, and is exponentially distributed for the regular ones. The inflow of searchers is such that 80% of all parking bays are occupied by regular motorists, and another 5% by special-needs motorists.

We will consider two policies. The exclusive policy, introduced in section 7.5, makes type-2 parking available only to special-needs searchers, i.e. $k_1 = \infty$ and therefore $D_x(k_1) = D, D_y(k_1, \infty) = 0$. Then, according to (31), $r_x(\infty, S_1) = S_1 - D$ and thus $r_n = r_y - r_x = 1 - S_1 - A_n t_n$. For both quantities to be positive, S_1 should belong to $(D, 1 - A_n t_n)$, which is $(0.8, 0.95)$ in our example. The optimality condition for S_1, (32), is then given by (34) with a unique solution at $S_1 = 89.9\%$ in our example.

The optimal policy of minimizing (30) with respect to S_1 and k_1 yields quite different results. Optimal S_1 turns out to be 85.2%, i.e. the share S_2 of special-needs parking is

Parameter Verbal description	Math description	Exclusive policy	Optimal policy
Share of special-needs parking, %	S_2	10.1	14.8
Type-1 "regular" occupancy, %	q_1	89.0	89.1
Type-2 "special needs" occupancy, %	q_2	49.5	61.5
Share of group-x among regular searchers, %	$\frac{\int_{\frac{c}{t}<k_1} a(c,t) \mathrm{d}c,t}{\int_{c,t} a(c,t) \mathrm{d}c,t}$	100	52.85
Group-x search duration, sec	$\frac{1}{r_x}$	24	26
Group-y search duration, sec	$\frac{1}{r_y}$	N/A	16
Special-needs group search duration, sec	$\frac{1}{r_n}$	47	42
Type-2 parking premium for regular searchers, \$ per hour	L_0	∞	0.674
Average social cost per parking bay, \$ per hour	(30)	0.587	0.557

Table 2: Steady states under exclusive and optimal regulation of special-needs parking

increased by approximately 50%. At the same time, optimal k_1 is low enough so that those with parking duration less than $\frac{c}{k_1} = 38$ minutes should be allowed to use special-needs parking. Table 2 compares the two policies.

Several results are worth noting. Under both policies, the special needs group has the highest duration of search, despite having very high cost of such search and despite lower occupancy of special-needs parking. This is because special-needs parking is rare and thus takes more time to find. Consistently with proposition 6, the optimal policy benefits the special-needs group by decreasing the duration of search by 10%. Essentially, optimal policy benefits everyone except the long-term "group-x" regular searchers who remain excluded from type-2 parking and face a smaller supply of parking bays. The aggregate objective function, i.e. expected search cost of all motorists plus the additional cost of special-needs parking supply, is decreased by about 5% by the optimal policy.

8. Conclusion

This paper is perhaps the first study that focuses on the economic benefits of discrimination in supply of parking. The main takeaway message is that discrimination can be useful in many cases; at the same time, optimal discrimination is less discriminating than many existing privileged parking policies, such as loading zones, handicapped parking, and

EV parking, which squarely exclude everyone except a minority of privileged motorists. In particular, the calibrated example of section 7 demonstrates that admission of non-privileged short term parkers to special-needs (e.g. handicapped) parking bays benefits not only the newly admitted, but even the original users, because such reform will be complemented by an increase in supply of special-needs parking.

In the environment where all parking is technologically identical, discrimination that excludes some searchers from some "privileged" vacant bays may be optimal as long as newly arriving motorists can steer their search towards most appropriate type of parking. Such discrimination, by partially segregating short-term or high-search-cost motorists, decreases social search cost by about 10% in calibrated examples. To maximize the ability of motorists to steer their search, privileged and non-privileged parking are best be located on different blocks. In case of Manhattan, for example, parking at avenues can be made more expensive than parking at streets.

Appendix A. Proof of lemma 1

The relationship between k_2 and δ can be found by calculating the mass of motorists parked at type-1 bays. On the one hand, such mass by definition is equal to $S_1 q_1 = S_1(D+\delta)$. On the other, this mass consists of two groups: a fraction of group-y and a fraction of group-z searchers. To calculate the former fraction, observe that a group-y searcher finds type-1 vacancy after each draw with probability $\bar{s}(1-q_1)$ and type-2 vacancy with probability $(1-\bar{s})(1-q_2)$, hence the fraction of group-y searchers who end up in type-1 parking is $\frac{\bar{s}(1-q_1)}{\bar{s}(1-q_1)+(1-\bar{s})(1-q_2)}$. Given the fact that both q_1 and q_2 deviate from D by an infinitesimal amount, this fraction can be approximated as $\bar{s}\left(1-\frac{1-\bar{s}}{S_2}\frac{\delta}{1-D}\right)$. By analogy with group y, the fraction of group-z searchers who end up in type-1 parking is equal to $\frac{\underline{s}(1-q_1)}{\underline{s}(1-q_1)+(1-\underline{s})(1-q_2)}$, which can be approximated by $\underline{s}\left(1-\frac{1-\underline{s}}{S_2}\frac{\delta}{1-D}\right)$.

To summarize, the following approximation for the mass of occupied type-1 parking is true:

$$S_1(D+\delta) = D_y(0,k_2)\bar{s}\left(1 - \frac{1-\bar{s}}{S_2}\frac{\delta}{1-D}\right) + D_z(0,k_2)\underline{s}\left(1 - \frac{1-\underline{s}}{S_2}\frac{\delta}{1-D}\right) + o(\delta).$$

By totally differentiating the above approximation with respect to k_2 and δ at point $\{\hat{k}_2, 0\}$ and rearranging, referring to appendix Appendix B for derivatives of D_x and D_y, and recalling (20), we obtain

$$\mathrm{d}\delta\left[S_1 + \frac{D}{1-D}\frac{1}{(\bar{s}-\underline{s})S_2}\left((S_1-\underline{s})\bar{s}(1-\bar{s}) + (\bar{s}-S_1)\underline{s}(1-\underline{s})\right)\right] = H(\hat{k}_2)(\bar{s}-\underline{s})\mathrm{d}k_2.$$

As both sides of the above are positive and finite, so is the derivative $\frac{\mathrm{d}\delta}{k_2}$ at point $\hat{k}_2$.

Appendix B. The derivatives

The derivatives of functions $D_w, G_w, w \in \{x,y,z\}$ defined in section 3.6 are as follows:

$$\frac{\mathrm{d}D_x}{\mathrm{d}k_1} = \int_t t^2 a(k_1 t, t)\mathrm{d}t \equiv H(k_1), \frac{\mathrm{d}D_y}{\mathrm{d}k_1} = -H(k_1), \frac{\mathrm{d}D_y}{\mathrm{d}k_2} = H(k_2) = -\frac{\mathrm{d}D_z}{\mathrm{d}k_2},$$
$$\frac{\mathrm{d}G_x}{\mathrm{d}k_1} = k_1 H(k_1) = -\frac{\mathrm{d}G_y}{\mathrm{d}k_1}, \frac{\mathrm{d}G_y}{\mathrm{d}k_2} = k_2 H(k_2) = -\frac{\mathrm{d}G_z}{\mathrm{d}k_2}.$$

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