Optimal Allocation of Multi-Dimensional Prizes in Contests with Heterogeneous Agents: Theory and an Empirical Application

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Abstract

I develop a model where two players with asymmetric preferences engage in a contest game. Prizes consist of two non-tradable divisible goods. I characterize the optimal prize allocation that maximizes aggregate effort of the contestants. When heterogeneity is severe, the designer benefits by giving a positive prize to the loser. This allocation eliminates the advantage of the stronger competitor and makes the contest homogeneous. As a consequence, the opponent has higher chances to win and exerts more effort in equilibrium. This positive response increases aggregate effort. The model mirrors the job promotion setting with monetary and non-monetary rewards. Using data from first-round matches of two professional tennis competitions where prizes include money and the ATP ranking points (career concerns), I structurally estimate contestants' skills and preferences. Overlooking multi-dimensionality results in biased estimates of the prize incentive effects. Counterfactual experiments show that reallocating 5% of money and 2% of the ATP ranking points from final winning prizes to first-round losing rewards could improve expected aggregate effort in relatively heterogeneous matches by more than 4.9%.

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1 Introduction

Many interactions can be characterized as bilateral contests with heterogeneous players and prizes consisting of multiple goods. The most important example is job promotion. Managers, who care about aggregate effort, can use not only monetary but also symbolic awards to motivate workers. The latter element of the prize schedule includes status or titles such as “Employee of The Month”. Often companies allocate firm-specific rights and privileges in the contest setting. By definition, there is no way to express non-monetary benefits in monetary terms. Thus, contest managers allocate bundles of prizes.

Workers can have heterogeneous preferences over reward dimensions. Some individuals may hold stronger career concerns but value monetary benefits less than their peers. As a result, the same prize schedule can shape contestants’ incentives differently. Managers must take this aspect into account when they allocate goods between bundles.

Overall, heterogeneity can worsen the contest performance significantly. When the competition is uneven, participants have fewer incentives to exert effort. Disadvantaged contestants know that most likely they will lose, and do not compete much. On the other hand, stronger opponents can make less effort than they would exert in the homogeneous contest, but still win. If the manager is not able to pre-select competitors, he may end up with very uneven matches and, consequently, low aggregate effort. In this respect, it is important to find a way to mitigate the detrimental effect heterogeneity has on players’ incentives.

Obviously, the contest needs sufficient winning prizes that provide enough stimuli to compete. At the same time, however, positive rewards for the weakest performers are not rare. For example, non-zero losing prizes are widespread in professional sports, which do not differ much from the job promotion setting. Each tournament of the Grand Slam series, a highly prestigious tennis competition, spends more than 10% of its total budget (about $4.5 million in 2015) to reward first-round losers. Perhaps surprisingly, also in this case prizes consist of two goods: money and the ATP ranking points (career concerns).

Based on these considerations, I study the optimal prize allocation in contests with heterogeneous participants and multiple reward items. In particular, I propose a theoretical model with two players characterized by asymmetric commonly known preferences. The designer has endowments of two non-tradable divisible goods. He can allocate the items between winning and losing bundles to maximize the total effort.

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1For example, see Lazear and Rosen (1981), Che and Gale (1998), Brown (2011).

2This series includes four competitions: the Australian Open, the French Open, the Wimbledon Championships, and the U.S. Open. The amount spent for first-round losing prizes exceeds the final winning prize by more than 50%.
When heterogeneity is strong, the optimal reward schedule prescribes a positive losing prize in the dimension valued by the advantaged participant relatively more. This scheme equalizes players’ winning benefits and makes the contest homogeneous. In other words, the designer reduces the advantage of the strong player, and the chances of the weaker contestant winning increase. As a result, the latter participant exerts more effort than he would in case of the “Winner–Takes–All” prize allocation. This positive effect is indirect and works through the equilibrium interaction (equilibrium effect). If heterogeneity is severe, higher losing prizes reduce the winning benefit of the advantaged player significantly, but do not affect his opponent much. Then the positive equilibrium effect dominates effort losses associated with lower winning benefits. Consequently, the total effort increases. This result contrasts with the previous findings of the contest design literature.

The described mechanism never works in the one-dimensional setting. To make the competition homogeneous, the designer must assign identical single-item prizes for a winner and a loser. The allocation, however, destroys the contest: both participants exert zero effort. In the two-dimensional case, the designer not only makes the competition even but also provides strictly positive winning benefits. The result is robust to heterogeneity in skills and asymmetric information about players’ types.

To stress the relevance of multi-dimensional prizes in job promotion and other similar contest settings, I collect data from first-round matches of two professional tennis competitions, the Australian Open and the French Open. The application is very close to the job promotion context. In particular, the athletes have two-dimensional incentives shaped by monetary prizes and career concerns (the ATP ranking points). At the same time, they show enough heterogeneity in skills and preferences. First-round winning prizes are approximated as a continuation value of being advanced in the tournament. I structurally estimate players’ skills and valuations. Importantly, both prize items matter for individual effort choices. When the points dimension is neglected, the model overestimates incentive effects of monetary prizes and underestimates contestants’ heterogeneity. This finding reveals the direction to improve the existing reduced-form tests of the contest theory that, to the best of my knowledge, always overlook the multi-dimensionality aspect.

Finally, I run counterfactual experiments and show how alternative reward schemes can improve aggregate effort in both tournaments. On average, the French Open features more heterogeneous matches and would benefit from larger than actual monetary losing prizes. However, the Australian Open could achieve the highest mean expected aggregate effort with zero monetary losing prizes. The observed difference can be explained with the contest-specific matching policy. If the managers

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3Grand Slam events are single-elimination tournaments consisting of seven stages: four rounds, a quarterfinal, a semifinal, and a final.

4For example, see Maloney and McCormick (2000), Sunde (2009), Brown (2013), Berger and Nicken (2014).
of analyzed tournaments could change the ranking points allocation, they would prefer zero losing prizes in this good. At the match-specific level, when the competition is relatively heterogeneous, reducing final winning trophies in money and the ATP ranking points in favor of first-round losing rewards by 5% and 2%, respectively, could increase expected aggregate effort by more than 4.9%. Compared to the “Winner–Takes–All” allocation, the gain would exceed 5.6%. On the other hand, relatively homogeneous matches never benefit from positive losing prizes. This counterfactual analysis is also relevant to a more general job promotion setting with fixed prize budgets and similar skill–valuation profiles.

The remaining of the paper is structured as follows. Section 2 inspects the related literature and stresses the contribution of this work. Section 3 characterizes the model and states main theoretical results. Section 4 describes the data and the structural setup, provides estimation results and counterfactual experiments. Section 5 concludes.

2 Related Literature

The paper contributes to several research fields. First, it is closely related to the contest design literature that focuses on the optimal prize allocation in different environments. Second, the current paper also complements the empirical work testing the contest theory predictions. To the best of my knowledge, both fields never looked at the setting with multi-dimensional prizes and heterogeneous preferences over them.

My theoretical contribution can be summarized as follows. Lazear and Rosen (1981) analyzed a tournament with two players and one-dimensional rewards. They showed that managers who want to maximize aggregate effort must always assign the highest feasible difference between winning and losing prizes. On the contrary, this paper shows when large prize spreads in all reward dimensions lower the total effort. Barut and Kovenock (1998), Moldovanu and Sela (2001) analyzed contests with more than two ex-ante symmetric participants and a single perfectly divisible good. Both papers found it optimal to assign up to \( N - 1 \) positive prizes for \( N \) contestants if the cost function is convex; otherwise, a winner must take all. This implicitly says that the designer who wants to induce the highest total effort in the single-good setting must never reward the weakest performer. In my model with two-dimensional prizes and ex-ante asymmetric contestants, higher losing benefits increase expected aggregate effort if heterogeneity in preferences and / or skills is severe. Importantly, this result does not rely on convexity of the effort cost function.

The models that investigate the effect of monetary and status incentives in contests show some similarities with the proposed multi-dimensional setting. Moldovanu et al. (2007) introduce a framework where ex-ante symmetric agents engage in a contest game, and the designer allocates

\[ Sisak (2008) \] provides a profound overview of the existing literature.
money or status in order to maximize expected aggregate effort. Importantly, the two prize dimensions are perfectly correlated. The authors show when it is optimal to implement a coarse partition into status groups, i.e. lump weaker performers together and give them the same losing prize. On the contrary, the top category must always include a single element to provide enough incentives to exert effort.

Dubey and Geanakoplos (2005) analyze contests where students with heterogeneous abilities compete for the prizes consisting of exam grades and the status derived from them. In this setting, coarse partitions result in more aggregate effort than grading on a curve because the former policy allows weaker participants to achieve top-ranks with a higher probability. The mechanism described in the current paper also incentivizes disadvantaged contestants to compete more. However, my setup, where no restriction on correlation is imposed, gives much more flexibility in the design-related aspects. For example, the schedules where a winner gets only one good and a loser takes another item can be optimal. This could not be the case in the aforementioned models with strongly correlated reward dimensions.

The paper also relates to the contest literature that focuses on the equilibrium characterization in the presence of consolation prizes. Baye et al. (2012) analyze symmetric Nash equilibria in all-pay auctions with two homogeneous players and different types of externalities. The latter feature allows a loser to obtain a positive payoff, which depends on the effort of his opponent, and makes a set of symmetric equilibria much larger than in a standard setting. In a very similar framework applied to a Tullock-type contest, Chowdhury and Sheremeta (2011) solve for a unique symmetric equilibrium. Both papers, however, do not address any design-related issues and take the prize allocation as given. On the contrary, the present work proves when positive losing benefits become optimal from the contest organizer’s prospective. Importantly, this result does not rely on externalities, but requires a sufficient degree of heterogeneity and multiple prize items.

Further, I highlight the contribution to the empirical literature on contests. Following standard theoretical models, this field does not address multi-dimensionality issues and consider only single-item rewards. The experimental evidence on prize incentive effects in contests is mixed. Most of the laboratory tests support the prediction that higher prize spreads induce more aggregate effort. At the same time, field experiments find conflicting evidence. For example, De Paola et al. (2016) introduce some elements of bilateral rank-order contests to the high-school grading system. They observe that growing prize spreads have no or a negative effect on students’ performance, which goes against the standard contest theory. However, the mechanism I describe in the current paper could explain this finding as follows. If students have multi-dimensional preferences (for example, they also care about status), but the designer is not aware of this, higher prize spreads in grades can reduce effort in relatively heterogeneous matches.

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6See Dechenaux, Kovenock, and Sheremeta (2015) for a review.
Non-experimental tests of the contest theory mainly deal with professional sports data. This setup is very close to the job promotion setting, but it has a clear advantage in terms of available proxies for contestants’ heterogeneity and effort choices. The vast majority of non-experimental tests use the reduced-form approach and follow the standard contest theory by looking at single-item prizes. Ehrenberg and Bagnano (1990), Maloney and McCormick (2000), Sunde (2003), Azmat and Moller (2008) test incentive effects of monetary rewards in running competitions and at the final stages of professional tennis tournaments. They find that higher monetary prize spreads increase effort and improve contestants’ performance, which is in line with the standard theoretical predictions. Sunde (2009), Brown (2011), Berger and Nicken (2014) confirm that professional athletes tend to exert less effort when the contest becomes more heterogeneous.

The current paper also employs data from professional sports, namely tennis. In contrast to the aforementioned reduced-form tests, I introduce two reward dimensions (i.e. money and the ATP ranking points) and structurally recover contestants’ skills and preferences. Although this estimation approach is neglected in the empirical contest literature, it has obvious advantages when one must restore utility parameters and run policy experiments. Importantly, overlooking the non-monetary dimension results in overestimated incentive effects of monetary prizes and underestimated heterogeneity. Similar to the reduced-form tests, I provide evidence that contestants tend to exert less effort in more uneven matches. Higher prize spreads, however, reduce expected aggregate effort in relatively heterogeneous competitions, and this contrasts with the previous empirical tests of the contest theory.

3 The Model

3.1 Model Setup

I analyze a complete information contest with two heterogeneous participants. This setting characterizes many interactions. Often job promotion contests for top positions or lobbying games do not have more than two participants. In professional sports such as tennis, football, basketball etc. only two players or teams compete in a particular tournament round. Other reasons I restrict the model in this way are tractability and exposition clarity.

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7Structural estimation is widely used in the applied auction and industrial organization literature. To the best of my knowledge, however, this is the first empirical test of the contest theory that structurally estimates the incentive effects of prize schedules.

8In this section, I do not introduce private information to keep the analysis as simple as possible but relax this assumption later on. Moreover, in many contest-type settings participants know each other’s characteristics. Often workers willing to be promoted are colleagues who have enough information about the opponent. In professional tennis, players observe competitors rankings, have some information about their training conditions, injuries etc.
The designer has two non-tradable perfectly divisible goods, $A$ and $B$, and there is no market to exchange the items.\footnote{If this is not the case, one goes back to a standard contest model with one-dimensional prizes.} Without loss of generality, I assume equal endowments of goods and normalize them to a unity: $\hat{A} = \hat{B} = 1$. Let $W = \begin{pmatrix} A^W \\ B^W \end{pmatrix}$ and $L = \begin{pmatrix} A^L \\ B^L \end{pmatrix}$ be winning and losing bundles, respectively.

Contestant $i$ attaches valuation $\alpha_i (\beta_i)$ to good $A (B)$. Players’ preferences are characterized by risk-neutrality and perfect additive separability:

\[ U_i (A^k, B^k) = \alpha_i A^k + \beta_i B^k \equiv U^k_i, \ i = \{1, 2\}, \ k = \{W, L\} \]

Competitors choose non-negative effort, $e_i$, simultaneously and pay a cost $\gamma_i (e_i) = e_i$.\footnote{For now, I impose no heterogeneity in contestants’ abilities but relax this assumption later on.} Player $i$ wins if he exerts more effort than the opponent, and ties are broken randomly. Then, if contestant $i$ obtains prize $k$, his payoff looks as follows:

\[ \pi_i (A^k, B^k, e_i) = U_i (A^k, B^k) - e_i \equiv \pi^k_i \]

Given the reward schedule and the competitor’s strategy, player $i$ chooses his effort in order to maximize the expected payoff:

\[ \max_{e_i} \left\{ P_{\{e_i > e_{-i}\}} U_i (A^W, B^W) + (1 - P_{\{e_i > e_{-i}\}}) U_i (A^L, B^L) - e_i \right\} \]

where $P_{\{e_i > e_{-i}\}} \in [0, 1]$ is a probability that player $i$ wins. Here I allow for both pure and mixed strategy equilibria in the game between contestants.

The designer observes only players’ relative standing and chooses the prize allocation that maximizes contestants’ aggregate effort. The game proceeds as follows:

1. The designer assigns $W$ and $L$ and commits to this reward schedule.
2. Contestants select $e_i, i = \{1, 2\}$ taking the prize allocation as given.

### 3.2 Contestants’ Equilibrium Behavior

Further, I analyzing contestants’ equilibrium behavior. Let $\Delta_A = A^W - A^L$ and $\Delta_B = B^W - B^L$ be prize spreads in corresponding reward dimensions. To avoid confusions with the case of heterogeneous abilities, I do not label players as “strong” and “weak”. Instead, contestants’ types are defined as follows:

**Definition 1.** Let $t_i (\Delta_A, \Delta_B) = U^W_i - U^L_i \equiv t_i$ be contestant $i$’s winning benefit, and define $t_g = \max \{t_1, t_2\}$, $t_n = \min \{t_1, t_2\}$. Then, $i$ is the “greediest-to-win” ($g = i$) if and only if $t_i = t_g$; otherwise, $i$ is not the “greediest-to-win” ($n = i$).
With this formulation, players’ types depend on the prize allocation. Then, the optimization problem of contestant $i$ can be rewritten in terms of his type:

$$\max_{e_i} \{ P(e_i > e_{-i})t_i + U^L_i - e_i \}$$

Since $t_g$ and $t_n$ are functions of contestants’ valuations and the reward schedule, types can be either positive or negative, and this will affect a structure of the equilibrium played.

**Definition 2.** The equilibrium is trivial if and only if at least one player chooses $e = 0$ with probability 1. Otherwise, the equilibrium is non-trivial.

Proposition 1 shows when there exists a non-trivial equilibrium and characterizes contestants’ strategies for different realizations of $t_n$:

**Proposition 1.** For $t_n \leq 0$ the equilibrium is always trivial. For $t_n > 0$ the equilibrium is unique and non-trivial:

- Contestant $g$ randomizes uniformly on $[0, t_n]$, and his equilibrium payoff is $\pi_g = t_g - t_n + U^L_g \geq U^L_g$.

- Contestant $n$ randomizes uniformly on $(0, t_n]$, places the atom of size $p^0_n = \frac{t_n - t_g}{t_g}$ at $e = 0$, and his equilibrium payoff is $\pi_n = U^L_n \leq \pi_g$.

**Proof.** See Appendix E. □

The non-trivial equilibrium from Proposition 1 is similar to the one Baye et al. (1996) derive for asymmetric all-pay auctions with two bidders and complete information. However, their characterization does not extend to the case of perfectly divisible goods and heterogeneous losing benefits.\(^{11}\) Also, Baye et al. (1996) cannot accommodate prize-dependent types. Hence, Proposition 1 provides a more general characterization than the aforementioned work.\(^{12}\)

As Proposition 1 states, two equilibrium configurations can emerge. When there is no strictly positive winning benefit for type $n$ ($t_n \leq 0$), this player always prefers to stay inactive, i.e. $e_n = 0$. If type $n$ deviates towards $e_n > 0$ and wins, his payoff becomes $\pi_n(e_n) = t_n + U^L_n - e_n$. However, $\pi_n(e_n)$ is strictly dominated by $\pi_n(0) = U^L_n$ for any $e_n > 0$. This equilibrium corresponds to the worst possible scenario from the designer’s prospective. On one hand, type $n$ never exerts positive effort. On the other hand, the lack of competition makes winning easier for player $g$ and drives $e_g$ down. As a result, the trivial equilibrium leads to the lowest aggregate effort possible.

\(^{11}\)In a single-object auction setting, players with lower bids always get nothing
\(^{12}\)The Baye et al. (1996) case is nested in the current equilibrium characterization. If one takes a one-dimensional reward schedule and fixes a winning prize at a unity, he gets exactly the same setup as Baye et al. (1996).
On the contrary, if type $n$ has incentives to compete ($t_n > 0$), both players choose $e > 0$ with a strictly positive probability. Then, contestants use mixed strategies, and ties happen with a zero probability. Importantly, type $n$ can never get higher equilibrium payoffs than his advantaged competitor. Since player $n$ does not exert effort that exceeds his type, the greediest participant can always choose $e_g \in [t_n, t_g)$ and win with certainty.\footnote{If type $n$ chooses $e_n = t_n + \varepsilon$, $\varepsilon > 0$ and wins, he gets $\pi_n(e_n) = U_n^L - \varepsilon$, and this is dominated by $e_n = t_n$.} Contestants’ equilibrium payoffs are equal if and only if the types coincide ($t_n = t_g$) and no positive losing prizes are assigned.

One could also interpret the atom contestant $n$ places at zero ($p_0^n$) as a relative power of player $g$. When $p_0^n \to 1$, the greediest type is extremely strong, and this destroys his opponent’s incentives to compete. If type $g$ has zero relative power, i.e. $p_0^g = 0$, the contest is equivalent to the homogeneous one.

Using Proposition 1, I can write down contestants’ expected effort in a closed form:

$$E_g = \frac{t_n}{2}, E_n = (1 - p_0^n) \frac{t_n}{2}$$

To emphasize the difference between one- and two-dimensional prizes, assume good $A$ is not available. In this case, type $g$’s relative power looks as follows:

$$p_0^g = \frac{\beta_g - \beta_n}{\beta_g} \forall \Delta B$$

and this value is constant. In other words, no matter how high stakes are, type $n$ stays inactive with the same probability. Thus, the only element of the expected effort that depends on $\Delta B$ is $t_n$, the winning benefit of contestant $n$. Since $\frac{\partial p_0^n}{\partial \Delta B} = \beta_n > 0$, lower prize spreads decrease expected effort of both contestants. Hence, one can already anticipate that with one-dimensional rewards the designer will always implement the “Winner–Takes–All” allocation. This result is well-known in the contest design literature.

Now suppose both dimensions become available. Then, the relative power of player $g$ depends on prize spreads, $\Delta_A$ and $\Delta_B$:

$$\frac{\partial p_0^g}{\partial \Delta_A} = - \left( \alpha_n \beta_g - \alpha_g \beta_n \right) \frac{\Delta_B}{t_g^2}$$

$$\frac{\partial p_0^g}{\partial \Delta_B} = \left( \alpha_n \beta_g - \alpha_g \beta_n \right) \frac{\Delta_A}{t_g^2}$$

When contestants have unequal marginal rates of substitution ($\frac{\alpha_n}{\beta_n} \neq \frac{\alpha_g}{\beta_g}$), these derivatives are typically of different signs.\footnote{If the latter condition is violated, type $g$’s relative power does not respond to changes in the prize allocation.} For example, take the “Winner–Takes–All” prize allocation ($\Delta_A = \Delta_B = 1$). If contestant $g$ is more sensitive to incentives in dimension $A$ than the opponent, his relative power must decrease (increase) in $\Delta_B$ ($\Delta_A$):

$$\Delta_A = 1, \frac{\alpha_n}{\beta_n} > \frac{\alpha_g}{\beta_g} \Rightarrow \frac{\partial p_0^g}{\partial \Delta_A} > 0, \frac{\partial p_0^g}{\partial \Delta_B} < 0$$
Next, suppose the designer reduces the prize spread in dimension $A$. First of all, this policy drives winning benefits of both players ($t_g$ and $t_n$) down, and they have less incentives to exert effort (direct effect). On the other hand, lower prize spreads in dimension $A$ undermine type $g$’s relative power and make the contest more even. As a result, player $n$ is incentivized to exert higher effort (equilibrium effect). When the latter positive effect dominates the negative one induced by the reduction of winning benefits, expected aggregate effort raises. I derive necessary and sufficient conditions for this to hold when characterize the optimal prize allocation.

### 3.3 The Optimal Prize Allocation

The designer chooses the prize allocation that maximizes expected aggregate effort and does not violate feasibility constraints:

$$J = \max_{A^W, A^L, B^W, B^L} [E_g + E_n]$$

s.t. $A^k, B^k \geq 0, k = \{W, L\}$

$$A^W + A^L \leq 1, B^W + B^L \leq 1$$

where $E_g = \frac{\alpha_n \Delta_A + \beta_n \Delta_B}{2}$ and $E_n = \left(\frac{\alpha_g \Delta_A + \beta_g \Delta_B}{\alpha_g \Delta_A + \beta_g \Delta_B}\right) \frac{\alpha_n \Delta_A + \beta_n \Delta_B}{2}$ are contestants’ expected effort.

#### 3.3.1 Fixed Prize Allocation in One Dimension

I start from the case when prizes in one dimension are fixed, i.e. the designer optimizes only over a particular good. Often contest managers act under similar constraints. Executives in organizations can decide about monetary rewards but must take the hierarchical structure of the company (workers’ status concerns) as given. In professional tennis, where money and ranking points are distributed, managers have no power to change prizes in the latter dimension because they are fixed by the ATP. Also, this restricted case is easier to analyze, and I use it as a building block for the main theoretical result later on.

Without loss of generality, assume that prizes in dimension $B$ are fixed:

$$B^W = \hat{B}^W, B^L = \hat{B}^L \Rightarrow \Delta_B = \hat{B}^W - \hat{B}^L$$

and the designer optimizes only over dimension $A$ including a possibility to leave good $B$ out. Let $r = \{\alpha_g, \alpha_n, \beta_g, \beta_n\}$ be contestants’ valuation profile, and $R$ denotes a set of all feasible $r$’s:

$$R = \{r : \alpha_i \geq 0, \beta_i \geq 0, i = \{g, n\}\}$$

\(^{15}\)Formally, $E_g$ has only one term, $t_n$, depending positively on the prize spreads. Hence, type $g$’s effort never grows when $\Delta_A$ becomes lower. However, player $n$’s expected effort includes two elements, $(1 - p^0_n)$ and $t_n$, that change in opposite directions when $\Delta_A$ decreases. If growth in $(1 - p^0_n)$ over-compensates losses in expected effort caused by lower winning benefits, $E_n$ increases.
The optimal prize allocation depends on contestants’ preferences. Proposition 2 characterizes all possible choices the designer can make given the valuation profile:

**Proposition 2.** For any \( \hat{\Delta}_B \in [-1, 1] \) and valuation profile \( r \in R \) the designer

- Uses both goods and leaves a positive losing prize in dimension \( A \), or
- Uses both goods and gives the endowment of \( A \) to a winner, or
- Does not use dimension \( B \) and gives the endowment of \( A \) to a winner.

**Proof.** See Appendix E.  

In the proof, I study properties of the designer’s objective function, \( J(\cdot) \), that can be expressed in terms of \( \Delta_A \) and \( \hat{\Delta}_B \). First, \( J(\cdot) \) is strictly convex in \( \Delta_A \), and the derivative of \( J(\cdot) \) with respect to \( \Delta_A \) is discontinuous at \( \hat{\Delta}_A = \frac{\beta_n}{\alpha_n - \alpha_n} \hat{\Delta}_B \). When the prize spread in dimension \( A \) is equal to \( \hat{\Delta}_A \), both contestants get identical winning benefits (namely, \( t_g = t_n \)). Let \( \hat{\Delta}_A^i, i = \{1, 2\} \) denote the allocation of item \( A \) that results in zero winning benefits for player \( i \):

\[
\hat{\Delta}_A^i = \left\{ \Delta_A : \alpha_i \Delta_A + \beta_i \hat{\Delta}_B = 0 \right\}
\]

Choosing \( \Delta_A \leq \max\{\hat{\Delta}_A^1, \hat{\Delta}_A^2\} \) (\( \Delta_A > \max\{\hat{\Delta}_A^1, \hat{\Delta}_A^2\} \)), the designer induces zero (positive) expected aggregate effort (see Proposition 1). If \( \hat{\Delta}_A \leq \max\{\hat{\Delta}_A^1, \hat{\Delta}_A^2\} \), the objective function \( J(\cdot) \) is strictly increasing in \( \Delta_A \in [\max\{\hat{\Delta}_A^1, \hat{\Delta}_A^2\}, 1] \). In this case, the designer must give the endowment of good \( A \) to a winner. Under \( \hat{\Delta}_A > \max\{\hat{\Delta}_A^1, \hat{\Delta}_A^2\} \), expected aggregate effort is strictly increasing in \( \Delta_A \) for any \( \Delta_A \in [\max\{\hat{\Delta}_A^1, \hat{\Delta}_A^2\}, \hat{\Delta}_A]) \), but can be non-monotone on the interval \( \Delta_A \in [\hat{\Delta}_A, 1] \). Given that \( J(\cdot) \) is strictly convex, there is no interior maximum in \( \Delta_A \in (\hat{\Delta}_A, 1) \). Thus, the two candidate prize bundles: \( (\hat{\Delta}_A, \hat{\Delta}_B) \) and \( (1, \hat{\Delta}_B) \) must be compared directly. In the proof of Proposition 2, I characterize valuation profiles for which \( (\hat{\Delta}_A, \hat{\Delta}_B) \) is preferred to \( (1, \hat{\Delta}_B) \), i.e. when giving a positive prize to a loser improves expected aggregate effort. Finally, since the designer cannot change the allocation of item \( B \) but is allowed to leave it out, he must directly compare expected aggregate effort induced by the best bundle and its counterpart in a single-good contest over dimension \( A \). Importantly, there exist valuation profiles such that the designer prefers the latter option.

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16 When rewards in dimension \( A \) are fixed, the sets of valuation profiles and the optimal prize allocation look exactly the same if one replaces \( \alpha \) with \( \beta \) and \( A \) with \( B \) in the proof.

17 When prizes are fully flexible, I use Proposition 2 to find the best reward schedule in dimension \( A \) given the optimal allocation of item \( B \).

18 The designer’s objective function has a kink at \( \hat{\Delta}_A = \frac{\beta_n}{\alpha_n - \alpha_n} \hat{\Delta}_B \), and this point can be either feasible or infeasible.
Proposition 2 partitions a set of valuation profiles \((R)\) into three subsets:
\[
R = \hat{R}^L \cup \hat{R}^{WTA} \cup \hat{R}^{SG}
\]
where \(\hat{R}^L (\hat{R}^{WTA})\) corresponds to optimal bundles with positive (zero) losing prize, \(\hat{R}^{SG}\) stands for contests with single-item rewards in dimension \(A\). More precisely, the designer’s choice is driven by the degree of heterogeneity in players’ valuations. In the beginning, I summarize these findings with a simple diagram and then explain the intuition. Let \(H_\alpha = \alpha_g - \alpha_n (H_\beta = \beta_g - \beta_n)\) be the degree of heterogeneity between contestants in dimension \(A (B)\). Also, assume that type \(g\) values both goods more than his competitor \((\alpha_g > \alpha_n, \beta_g > \beta_n)\). Figure 1 shows how the designer’s choice depends on \(H_\alpha\) and \(H_\beta\) in case of \(\hat{\Delta}_B \geq 0\). I keep \(\alpha_g (\beta_g)\) constant and attribute all changes in \(H_\alpha (H_\beta)\) to \(\alpha_n (\beta_n)\).


diagram

Figure 1: Optimal Prize Allocation as a Function of Contestants’ Heterogeneity: \(\hat{\Delta}_B \geq 0\)

Note: \(H_\alpha = \alpha_g - \alpha_n (H_\beta = \beta_g - \beta_n)\) corresponds to the degree of heterogeneity between contestants in dimension \(A (B)\); \(\alpha_g (\beta_g)\) is kept constant, \(\alpha_g > \alpha_n\) and \(\beta_g > \beta_n\); values with asterisks denote thresholds imposed on the degree of contestants’ heterogeneity in different dimensions.

Although the prizes in good \(B\) are fixed, heterogeneity in both dimensions affects the designer’s choice. The “Winner–Takes–All” allocation is optimal when contestants have similar preferences over reward items. If heterogeneity in dimension \(A\) is severe, the designer prefers to leave a positive losing prize in good \(A\). Finally, when preferences over item \(A\) but not \(B\) are relatively homogeneous, neglecting the latter dimension results in the highest expected aggregate effort.

Optimal positive losing prizes in dimension \(A\) require enough asymmetry in contestants’ valuations. Importantly, player \(g\) must be more sensitive to incentives shaped by good \(A\) than his competitor:
\[
\begin{align*}
\alpha_g > \alpha_n \\
\frac{\alpha_n}{\beta_g} > \frac{\alpha_n}{\beta_n}
\end{align*}
\]

A sufficient condition to make \(A^L > 0\) optimal is strong heterogeneity in dimension \(A\).

Since contestants’ expected effort depends on the prize spreads, one can think about the equivalence of the following alternatives:

1. Use the endowment of good \(A\) completely and increase the losing prize at the cost of the winning reward (\(\Delta_A = \tilde{\Delta}_A < 1\) and \(A^W + A^L = 1\)), or

2. Waste some endowment of good \(A\), assign zero losing (winning) benefit in this item and give the winner (the loser) \(A^W = \tilde{\Delta}_A < 1\) in case of \(\tilde{\Delta}_A \geq 0\) (\(A^L = -\tilde{\Delta}_A < 1\) in case of \(\tilde{\Delta}_A \in (-1, 0)\)).

The designer, however, prefers option 1 (no waste) and does not destroy the endowment of item \(A\). Remarkably, when type \(g\) values both goods more and players’ preferences in dimension \(A\) differ a lot, the designer benefits by assigning \(\Delta_A < 0\), i.e. the prize spread becomes negative.\(^{19}\) The latter observation makes the result robust to the introduction of costly prizes: even then the designer will be willing to reward the weakest performer in case of severe heterogeneity.

Overall, positive losing prizes affect the effort players exert in two ways:

1. Since winning benefits decrease, both contestants have less incentives to fight. I call this **direct effect**.

2. When player \(i\) reduces his effort, the opponent is willing to compete more and, consequently, wins with a higher probability. I label this indirect response to growing losing prizes as **equilibrium effect**.

Suppose \(\tilde{\Delta}_A = \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n} \hat{\Delta}_B\) is feasible.\(^{20}\) If the designer implements the \((\tilde{\Delta}_A, \hat{\Delta}_B)\) allocation, players’ winning benefits (i.e. their types) look as follows:

\[
t_g = t_n = \frac{\alpha_g \beta_n - \alpha_n \beta_g}{\alpha_g - \alpha_n} \hat{\Delta}_B \equiv \tilde{t}
\]

and the contest becomes homogeneous. When type \(g\) prefers good \(A\) relatively more than his opponent (\(\alpha_g > \alpha_n\) combined with \(\frac{\alpha_n}{\beta_g} > \frac{\alpha_n}{\beta_n}\)) and \(\hat{\Delta}_B > 0\), the value of \(\tilde{t}\) is positive. Importantly, the increase in losing benefits does not destroy contestants’ incentives to compete. Moreover, the \((\tilde{\Delta}_A, \hat{\Delta}_B)\) allocation redistributes the relative power in favor of type \(n\):

\(^{19}\)In case of extreme heterogeneity in dimension \(A\), the designer prefers to give the endowment of this good to a loser.

\(^{20}\)When \(\tilde{\Delta}_A = \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n} \hat{\Delta}_B\) is infeasible, the designer assigns the lowest possible prize spread in dimension \(A\) (i.e. \(\Delta_A = -1\)).
Then, player \( n \) is more likely to succeed, and his incentives to compete increase compared to the “Winner–Takes–All” schedule. Thus, higher losing prizes reinforce the positive equilibrium effect.

In the one-dimensional setting, the designer can never make the contest homogeneous and at the same time provide sufficient stimuli to exert effort. Assume good \( B \) is not available:

\[
t_g = \alpha_g \Delta A, \; t_n = \alpha_n \Delta A
\]

To equalize contestants’ types, the designer must assign identical winning and losing prizes (i.e. \( \Delta A = 0 \)). This, however, destroys incentives completely, and in equilibrium, both participants exert zero effort. In the two-dimensional case, the designer not only makes the match even but also preserves strictly positive winning benefits. This mechanism still works when one introduces heterogeneity in skills and asymmetric information about players’ types to the model.\(^{21}\)

Another key ingredient that allows the designer to affect the relative power of type \( g \) is asymmetry in contestants’ valuations (i.e. \( \frac{\alpha_g}{\beta_g} \neq \frac{\alpha_n}{\beta_n} \)). The winning benefit of player \( g \) must decline faster than the one of his opponent when losing prizes grow.\(^{22}\) On the contrary, collinear preferences (\( \frac{\alpha_g}{\beta_g} = \frac{\alpha_n}{\beta_n} \)) will make the designer unable to redistribute the relative power without destroying incentives to compete (in this case, \( \bar{t} = 0 \) holds). The latter setting has same qualitative properties as the one-dimensional model where the “Winner–Takes–All” scheme maximizes expected aggregate effort.

Now, I characterize how heterogeneity affects the optimal prize allocation (see Figure 2). When contestants have similar preferences, player \( g \) does not have a significant advantage over the opponent. Further, winning benefits of both types (namely, \( g \) and \( n \)) decline more or less symmetrically when losing rewards grow (see Figure 2). In this case, the positive equilibrium effect will never compensate losses associated with the winning prize reduction (direct effect). As a result, the designer implements the “Winner–Takes–All” allocation and does not redistribute the relative power between competitors if heterogeneity is weak.

When the advantage of type \( g \) is strong, the opponent does not have enough incentives to compete. As a consequence, the “Winner–Takes–All” scheme cannot support the highest expected aggregate effort. On the contrary, positive losing prizes make players’ types more balanced (ideally, \( t_n = t_g \)), and the contest becomes homogeneous. Since preferences are asymmetric, the policy reduces winning benefits of the advantaged competitor significantly but does not affect the opponent much (see Figure 2). Thus, the positive equilibrium effect dominates the direct one, and higher losing prizes increase expected aggregate effort.

\(^{21}\)See Subsection 3.4 and Appendix A, respectively.

\(^{22}\)Since the reward structure in dimension \( B \) is fixed, all changes in winning benefits are driven by shifts in \( \Delta A \).
Finally, I comment on the preference profiles entering $\hat{R}^{SG}$. When the positive losing prize in item $A$ is optimal (i.e. $\Delta_A = \tilde{\Delta}_A$), the designer never ignores dimension $B$. Given that the $(\tilde{\Delta}_A, \tilde{\Delta}_B)$ allocation requires strong heterogeneity in preferences over good $A$, a single-item reward schedule in this dimension does not result in sufficient effort. However, with the second good at place, the designer can balance players’ incentives to compete and make the contest homogeneous.
If preferences over item $A$ are similar, the endowment of this good must be assigned to a winner ($\Delta_A = 1$). Given $\tilde{\Delta}_B$ and contestants’ valuations in dimension $B$, the designer must decide whether to use item $B$ or leave it out. In particular, for $\tilde{\Delta}_B \geq 0$ he prefers to run a single-good contest in dimension $A$ under following conditions:

1. Players respond to incentives in dimension $A$ the most ($\alpha_i \gg \beta_i$);
2. Contestant $g$ values both goods more than the opponent ($\alpha_g \geq \alpha_n$, $\beta_g > \beta_n$);
3. Heterogeneity in dimension $B$ ($A$) is sufficiently strong (weak), and contestant $g$ is more sensitive to changes in $B^W$ and $B^L$, i.e. $\frac{\beta_g}{\alpha_g} > \frac{\beta_n}{\alpha_n}$.

To highlight the intuition, take the case of relatively homogeneous preferences in dimension $A$ ($\alpha_g \approx \alpha_n$). When the designer uses item $B$, winning benefits grow (direct effect). At the same time, the competition becomes more heterogeneous: since $\beta_g > \beta_n$, the advantage of type $g$ rises. As a result, in equilibrium, player $n$ has less incentives to exert effort (namely, the atom type $n$ places at $e = 0$ increases). If heterogeneity in dimension $B$ is strong, the gap in contestants’ winning benefits increases significantly when item $B$ is at place, and the negative equilibrium effect prevails. If the designer could adjust the prizes in good $B$, he would prefer to assign higher losing benefits and make the two-dimensional competition even. Since it is not feasible, the designer uses only good $A$ and runs a homogeneous single-item contest. In this case, players still exert enough effort because they are mainly motivated by item $A$ and heterogeneity in this dimension is weak.

If contestant $g$ values only good $A$ more than the opponent ($\alpha_g \geq \alpha_n$ combined with $\beta_g < \beta_n$), the bundle is always optimal. The presence of item $B$ helps the designer to reduce the advantage of player $g$ in dimension $A$ and provide stronger incentives to compete (both in terms of direct and equilibrium effects).

### 3.3.2 Flexible Prize Allocation in Both Dimensions

In this subsection, I analyze the most general case when the designer can change the reward schedule in both dimensions. To characterize the optimal prize allocation, I use the results of Proposition 2. As before, $R$ denotes a set of all feasible valuation profiles. Theorem 1 characterizes the designer’s optimal choice given contestants’ preferences:

**Theorem 1.** For any valuation profile $r \in R$ the designer uses goods’ endowments completely and either leaves a positive losing prize at least in one dimension or gives both items to a winner.

**Proof.** See Appendix E. 

Theorem 1 employs the proof of Proposition 2 as a building block. I define the designer’s objective function, $J(\cdot)$, in terms of $\Delta_A$ and $\Delta_B$. Now, $J(\cdot)$ has kinks at $\tilde{\Delta}_A = \frac{\beta_n-\beta_g}{\alpha_g-\alpha_n} \Delta_B$ and
\[ \Delta_B = \frac{\alpha_g - \alpha_n}{\beta_g - \beta_n} \Delta_A. \] Also, \( J(\cdot) \) is strictly convex, and the designer’s problem has no interior solution. Then, fix the prize allocation in dimension \( B \), i.e. \( \Delta_B = \hat{\Delta}_B \). Proposition 2 implies that for any \( \hat{\Delta}_B \) there exists a non-empty set of valuation profiles such that the designer prefers to assign a positive losing prize in item \( A \). Then, the statement must hold for \( \hat{\Delta}_B = 1 \) as well, and for some valuation profiles the “Winner–Takes–All” allocation is not optimal.

Using the result of Theorem 1, one can partition a set of valuation profiles \( \tilde{R} \) into two subsets:

\[ R = R^L \cup R^{WTA} \]

where \( R^L \) (\( R^{WTA} \)) corresponds to bundles with positive (zero) losing prizes. Overall, properties of \( R^L \) and \( R^{WTA} \) are very similar to those of \( \hat{R}^L \) and \( \hat{R}^{WTA} \) from Proposition 2. However, with the fully flexible reward schedule, there do not exist valuation profiles such that it is optimal to ignore one good (recall \( \hat{R}^{SG} \) from Proposition 2). This fact is not surprising given that the corresponding result in Proposition 2 was driven by impossibility to adjust the rewards in dimension \( B \).

Figure 3 depicts the optimal prize allocation as a function of players’ heterogeneity. When the contest is relatively even, the “Winner–Takes–All” bundle induces the highest expected aggregate effort. If players show strong heterogeneity in the dimension where type \( g \)'s incentives are more sensitive (good \( A \) for \( \frac{\alpha_g}{\beta_g} > \frac{\alpha_n}{\beta_n} \) and \( B \), otherwise), it is optimal to assign a positive losing prize in the corresponding item.

Figure 3: Optimal Prize Allocation as a Function of Contestants’ Heterogeneity: Flexible Rewards in Two Dimensions

\[ \text{Note: } H_\alpha = \alpha_g - \alpha_n \quad (H_\beta = \beta_g - \beta_n) \text{ corresponds to the degree of heterogeneity between contestants in dimension } A \quad (B); \quad \alpha_g (\beta_g) \text{ is kept constant; } \alpha_g > \alpha_n \text{ and } \beta_g > \beta_n \text{; values with asterisks denote thresholds imposed on the degree of contestants’ heterogeneity in different dimensions.} \]

The mechanism driving the results was described in Subsection 3.3.1. On one hand, the positive

\[ \text{There also exist valuation structures such that it is optimal to give positive losing prizes in both items. Then the lowest prize spread must correspond to the dimension in which type } g \text{'s incentives are more sensitive.} \]
losing prize reduces winning benefits and, consequently, incentives to compete (direct effect). At the same time, it destroys the relative advantage of the “greediest” player \((g)\) that motivates the opponent to exert more effort (equilibrium effect). When contestants’ preferences are very heterogeneous, the latter effect dominates, and expected aggregate effort increases. To illustrate all the points made, I provide a simple numerical example:

**Example.** Suppose \(\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \end{pmatrix}\) and \(\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}\). The “Winner–Takes–All” schedule results in \(J(1, 0, 1, 0) \approx 14\). Player 1 is the “greediest-to-win” \((g = 1)\) because his winning benefit exceeds the one of the opponent. For given \(\alpha_g\) and \(\alpha_n\), it is always optimal to assign the highest prize spread in dimension \(A\) (see the proof of Theorem 1). Contestant \(g\) is more sensitive to incentives in dimension \(B\) \((\frac{\alpha_g}{\beta_g} < \frac{\alpha_n}{\beta_n})\). Hence, optimal positive losing prizes may appear only in the second good.

Now assume the designer implements another reward schedule in dimension \(B\): \(\begin{pmatrix} B_W \\ B_L \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{7} \end{pmatrix}\), and \(\Delta_B = \tilde{\Delta}_B = -\frac{1}{2}\). This results in \(J(1, 0, \frac{1}{4}, \frac{3}{4}) = 6\), and the “Winner–Takes–All” scheme dominates the latter allocation. In this case, the positive equilibrium effect is relatively small because contestants do not show enough heterogeneity in preferences over good \(B\).

Next, take the same \(\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}\) combined with \(\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}\). The “Winner–Takes–All” allocation results in \(J(1, 0, 1, 0) \approx 7.8\). As before, player 1 is the “greediest-to-win”, and positive losing prizes may appear only in dimension \(B\). The alternative reward scheme with \(\begin{pmatrix} B_W \\ B_L \end{pmatrix} = \begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \end{pmatrix}\), and \(\Delta_B = \tilde{\Delta}_B = -\frac{1}{7}\) induces \(J(1, 0, \frac{3}{7}, \frac{4}{7}) \approx 9\). As a result, the designer benefits from the positive losing prize in dimension \(B\). This is the case because contestants’ preferences over good \(B\) differ a lot, and the equilibrium effect prevails.

### 3.4 Heterogeneity in Skills

Before I assumed that contestants have identical effort costs, and the only source of heterogeneity stems from preferences. However, in most empirical applications, players differ in abilities, experience, and other characteristics that can shape their skills. To account for this fact, I introduce player-specific cost functions. This setup is used in Section 4 to estimate contestants’ skills and preferences.

Reformulate the model slightly. Assume contestant \(i\) has the following cost function:

\[ \gamma_i(e) = \frac{e}{c_i}, \quad c_i > 0 \]

where \(c_i\) is a skill parameter, \(i = \{1, 2\}\). Higher \(c_i\) makes effort exertion less costly. The expected payoff of player \(i\) looks as follows:

\[ P(e_i > e_j) U^W_i + (1 - P(e_i > e_j)) U^L_i - \frac{e}{c_i} \]
Applying a monotone transformation, one can rewrite the problem contestant \(i\) solves:
\[
\max_{e_i} \left\{ c_i \left[ P(e_i > e_j) U^W_i + (1 - P(e_i > e_j)) U^L_i \right] - e_i \right\}, \; i \neq j
\]

Now, winning benefits of both players depend on valuations and skills. Define \(\tilde{U}^W_i = c_i U^W_i, \tilde{U}^L_i = c_i U^L_i, \) and \(\tilde{t}_i = c_i (U^W_i - U^L_i)\). Replacing \(t_i\) with \(\tilde{t}_i\) brings us back to the original model. The equilibrium characterization can be reformulated in terms of skill-valuation profiles,
\[
\tilde{r} = \{\tilde{\alpha}_g, \tilde{\alpha}_n, \tilde{\beta}_g, \tilde{\beta}_n\}, \quad \text{where} \quad \tilde{\alpha}_i = c_i \alpha_i, \tilde{\beta}_i = c_i \beta_i.
\]

Let \(\tilde{R}\) be a set of all feasible \(\tilde{r}\)'s:
\[
\tilde{R} = \left\{ \tilde{r} : \tilde{\alpha}_i \geq 0, \tilde{\beta}_i \geq 0, i = \{g, n\} \right\}
\]

Proposition 3 characterizes the optimal prize allocation for given skill-valuation profiles:

**Proposition 3.** For any skill-valuation profile \(\tilde{r} \in \tilde{R}\) the designer uses goods’ endowments completely and either leaves a positive losing prize at least in one dimension or gives both items to a winner.

**Proof.** The new optimization problem is a rescaled version of the original program. The proof follows immediately when one replaces the elements of \(R\) with those of \(\tilde{R}\) in Theorem 1.

Proposition 3 mirrors Theorem 1 applied to the case of heterogeneous skills. As before, optimal positive losing prizes require asymmetry in contestants’ valuations and strong heterogeneity in the dimension preferred by player \(g\) relatively more. The mechanism driving this result was described in Subsection 3.3.1.

Skills can contribute to players’ heterogeneity in different ways. Let \(H^A (\tilde{H}^A)\) be a degree of heterogeneity over dimension \(A\) in the original model (the setup with different skills):
\[
H^A = \alpha_g - \alpha_n, \quad \tilde{H}^A = c_g \alpha_g - c_n \alpha_n
\]
Depending on \(c_g\) and \(c_n\), heterogeneous skills can mitigate or amplify asymmetries in contestants’ preferences.

This relatively simple model with no private information already helps in explaining a co-existence of positive winning and losing prizes. The results derived above could never be obtained in a standard single-item framework. Remarkably, optimality of non-zero losing benefits does not rely on convexity of the cost function as a traditional argument in favor of multiple positive prizes.\(^{24}\) However, even then it would never be optimal to reward the weakest performer in case of single-item rewards.

In Appendix A, I extend the original model in two ways. First, I allow the designer to run separate competitions over both reward dimensions instead of making the prize bundle. With this setting, players exert effort in contests \(A\) and \(B\) and get single-item rewards in corresponding items. Since prizes are costless for the designer, he can always unbundle goods unless specific constraints

\(^{24}\)For instance, see Moldovanu and Sela (2001), Barut and Kovenock (1998).
are imposed. If aggregate effort taken over two separate contests always dominates the one induce by bundling schemes, the results of Theorem 1 cannot be generalized and break down easily.

Theorem 2 shows that there exist preference profiles for which bundles (including those with positive losing prizes) generate strictly more expected aggregate effort than two simultaneous single-item contests. The intuition is as follows. Bundles with positive losing prizes become optimal when players’ preferences display strong heterogeneity. This also means that corresponding single-item contests are very uneven and result in low expected aggregate effort. Hence, the two-dimensional allocation with positive losing benefits helps the designer to mitigate a negative effect heterogeneity has on players’ incentives to exert effort.

Another extension introduces asymmetric information about contestants’ types. This modification is important for two reasons. First, players can have unobservable characteristics that affect their effort choices. Second, often the designer must commit to the prize allocation before he learns the exact matching. For example, tournament organizers in professional tennis announce monetary reward schedules before the final draw is known.

Suppose that contestants’ preferences over dimension $A$ ($\alpha_i$) constitute their private information. For simplicity, I assume that $\alpha_i$ has only two realizations: $\alpha_i = \{\bar{\alpha}_i, \tilde{\alpha}_i\}$ with $P(\alpha_i = \bar{\alpha}_i) = k$ and $\alpha_1, \alpha_2$ are drawn independently. I find that there exist probability-valuation profiles, defined over $k$ and contestants’ preferences, for which it is optimal to make a bundle with a positive losing prize. The underlying intuition is the same as described in Subsection 3.3.1.

4 The Empirical Application

This section estimates the model and introduces policy experiments. The most important application of the proposed theoretical framework is job promotion with monetary and non-monetary rewards. Unfortunately, there do not exist sufficient firm-level data that approximate contestants’ heterogeneity and their effort choices. Nevertheless, this information can be restored if one analyzes professional sport competitions. The setup does not differ much from the job promotion context. Athletes often have two-dimensional incentives shaped by monetary prizes and career concerns (their position in the ranking). At the same time, they show enough heterogeneity in skills. Thus, the results obtained with the sports data generalize to the job promotion and other similar settings.

To estimate the model, I use a sample of first-round games from two highly prestigious tennis tournaments, the Australian Open and the French Open. These contests display the key features of the theoretical model. Importantly, players care about two goods: prize money and their position in the ranking. Regarding the latter dimension, each year tennis players participate in a sequence
of tournaments and earn points. The rankings for males and females are updated and released every week.

Obviously, the indicated prize dimensions display positive correlation. However, there is no market to exchange the two goods of interest. One can argue the ATP ranking points matter for contestants’ effort exertion. In fact, this prize dimension maps into career concerns. Being the top player gives an easier access to other tournaments, media coverage, recognition, contracts for participation in advertising campaigns etc. Further, in the end of every game season top–8 male athletes are nominated to play in the ATP World Tour Finals, an extremely prestigious competition. Another argument stressing the relevance of the ranking is the existence of well-recognized tournaments with no money at stake. For instance, in 2009–2015 the Davis Cup allocated only the ATP points. Nevertheless, top–athletes such as Novak Djokovic, Andy Murray, Stan Wawrinka participated in the competition. Later on, I provide more evidence that the ATP ranking points matter for contestants’ effort exertion (see Subsection 4.2.2).

Tennis players show enough variation in skills. Moreover, their preferences over prize dimensions can differ too. Given the empirical distribution of ranking points (see Figure 5), 10 additional units of the good can never change positions of top–10 players. However, the same amount may move athletes ranked 90–100 up to 9 positions (keeping points of their closest peers constant). Finally, tennis players are professionals who perfectly know the rules of the game and act strategically. Thus, one should expect them to behave rationally in the given environment.

Importantly, contest organizers in professional tennis cannot change the allocation of ranking points between tournament stages. This element of the prize schedule is fixed by the ATP and the WTA for males and females, respectively. However, tournament managers have a power to vary monetary rewards. Very often, they leave positive prizes for first-round losers. For example, in 2013, the Australian Open increased this element of the reward schedule significantly. At the same time, other prizes did not evolve as much (see Tables 6 and 7). Both contest managers and the players expected this policy to induce more competition at earlier stages of the tournament. However, one could come with alternative explanations of the changes observed. The first argument in favor of relatively high losing stakes in first-round games would be a participation story. Nevertheless, this does not seem to be a single driving force in case of highly prestigious contests:

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25 The same scheme exists for females.
26 Starting from 2016, no points are assigned for Davis Cup ties.
As a rule, players get paid for their participation. This amount covers traveling costs and other supplementary expenses. However, the payment cannot be counted as a prize because it does not generate incentives to compete.
27 In this respect the proposed application has a clear advantage over experimental tests one can run in the lab where participants need time to learn rules of the game.
1. Four Grand Slam tournaments do not overlap in time. As a result, they do not need to compete for players. Moreover, the contests have very similar sets of participants and comparable prize schedules. In particular, the allocation of points is identical for all Grand Slams.

2. Top players who are likely to advance in the contest are always willing to participate. First, winning the Grand Slam gives 2,000 ranking points when other tournaments pay at most 1,000. Second, significant monetary rewards allocated in the finals induce both participation and effort exertion. Since best athletes win with a higher probability, they always prefer entering the competition to abstention.

3. Weaker athletes must compete in qualifying games to get one out of 16 places in the main draw of the Grand Slam. Thus, participation is competitive.

4. The Australian Open compensates traveling and living costs of the participants. Thus, positive monetary prizes for first-round losers cannot be seen as a way to reimburse these expenses.

Although the participation story is potentially important, it cannot completely explain how higher losing prizes affect effort choices.

4.1 The Data

I analyze the Australian Open and the French Open in 2009–2015. Only first-round matches for male players are taken into account. There are several reasons to restrict the dataset in such a way. I do not look at later stages of the contests because one can treat losing rewards assigned there as the prize for being advanced. However, I account for these prizes when compute first-round winning benefits (more details will follow). Second, in 2009 the ATP changed the ranking points allocation between stages of the Grand Slams. Thus, years before 2009 are excluded as an attempt to isolate possible structural breaks. Also, a relatively short horizon (2009–2015) allows me to assume stable preferences over time.

As mentioned before, in 2013, the Australian Open increased first-round monetary losing prizes significantly (up to 40% compared to the previous year). The policy might have shifted players’ preferences or made them contest-specific. Given that the Australian Open and the French Open

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29In fact, the time interval between the Australian Open and the nearest Grand Slam is about 3 months. This gives players enough time to recover and prepare for the next big competition.


31The data are available online at https://matchstat.com/.

32See Tables 6 and 7.
have similar sets of participants, I use the latter competition as a control group for the former one. Also adding another comparable tournament makes a set of observed prizes and players’ responses richer.

Another argument in favor of choosing the French Open but not another Grand Slam as a control group is the contest’s position in the annual tournament schedule. In fact, the French Open (end of May) follows the Australian Open (mid-January). Thus, both competitions take place in the first half of the game season. Given that the ATP World Tour Finals played by top-8 athletes are held in November, contestants’ preferences may change when the end of the season is approaching. If one takes the Wimbledon (late June) or the US Open (late August), stronger players willing to qualify as top-8 can shift their preferences in favor of non-monetary prizes. As a result, contestants’ behavior might evolve once the game season proceeds. In this respect, the French Open is the best control group to avoid inconsistencies in preference estimates.

I exclude female players from the sample for two reasons:

1. In 2014, the WTA changed the ranking points allocation between stages of the Grand Slams. The policy targeted only females. The event could shift players’ preferences, and for now I prefer to isolate this source of variation.

2. In Grand Slam events, females play according to the “best-of-3” scheme, but males follow the “best-of-5” rule. The two settings may shape the athletes’ behavior differently. Moreover, with the “best-of-3” scenario, the effect of luck or other random factors on the contest outcome might be stronger. Hence, it is potentially problematic to get consistent estimates of skills and preferences in the econometric model where females and males are analyzed together.

Further, one have to find a good proxy for contestants’ effort. Most statistics reported in tennis matches are relative. However, to estimate the model, absolute effort must be approximated. I solve this issue by constructing a measure based on a number of unforced errors:

“An unforced error is when the player has time to prepare and position himself or herself to get the ball back in play and makes an error. This is a shot that the player would normally get back into play.”

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33 Male and female professional tennis players belong to different associations. These two entities fix the points’ allocation over annual tournaments and calculate athletes’ rankings independently.

34 In Grand Slams, females and males play at most 3 and 5 sets per match, respectively.

35 For example, a number of points won in a particular match depends on the actions of both players.

By definition, the unforced error is not the outcome of a direct strategic interaction between competitors.\textsuperscript{37} Timothy W. Gallwey, a professional tennis coach, says in one of his books, “When I was playing tournament tennis and made an unforced error, I understood that the actual cause of my error was a momentary lapse in focus. I let the error be a wake-up call to come back to the present moment.”\textsuperscript{38} The claim relates unforced errors to lower effort exertion. One may argue such mistakes are driven by risk-taking or characterize a player-specific game style. These concerns are potentially important, and I will address this point later.

Let \( u_{ij} \) be a number of unforced errors player \( i \) makes in match \( j \) (see Table 8 for summary statistics). Since longer games are likely to show more unforced errors, I weight \( u_{ij} \) by a number of sets played \( (ns_j) \) and define \( \tilde{u}_{ij} = \frac{u_{ij}}{ns_j} \). Finally, assume there exists a continuous function \( h \) (identical for all contestants) that transforms individual effort \( (e_{ij}) \) into \( \tilde{u}_{ij} \), \( \frac{\partial h(e_{ij})}{e_{ij}} < 0 \). In other words, higher \( e_{ij} \) translates into less unforced errors per set played. For simplicity, take \( h(e_{ij}) = \hat{u} - \tilde{e}_{ij} \) where \( \hat{u} = \max_{i,j} \tilde{u}_{ij} \).\textsuperscript{39}

In Appendix B, I show that the proposed measure is a good proxy for effort. Standard contest models with one-dimensional prizes deliver two predictions:

1. Effort must increase in the prize spread.
2. Effort must decrease in contestants’ heterogeneity.

Reduced-form non-experimental tests of the contest theory conducted with different effort measures strongly support these hypotheses.\textsuperscript{40} The same holds for the proxy based on a number of unforced errors per set. This allows me to conclude that the current measure performs at least as good as the alternatives used in the previous work. Further, I investigate if the proxy includes a risk-taking component. To perform the test, I assume that contestants who choose riskier strategies tend to serve with a higher speed. Then, if the number of unforced errors per set also includes risk-taking, the two variables must be positively correlated. Table 17 shows that this hypothesis must be rejected.

Next, I check the properties of the points’ empirical distribution (see Figure 5). Only 1\% of players hold more than 10,000 ranking points. At the same time, around 70\% have 1,000 points or less. In fact, the distribution is very skewed towards the left. To capture this feature, players

\textsuperscript{37} Paserman (2010) also looks at a number of unforced errors to measure the performance of tennis players. He claims this approach makes an improvement over previous empirical studies on tennis that approximated effort as the number of games or sets won.


\textsuperscript{39} In principle, one could come with more complicated specifications of \( h \). However, given unobservability of effort, there would be no way to achieve identification. For this reason I stick to the simplest possible functional form that supports \( \frac{\partial h(e_{ij})}{e_{ij}} < 0 \).

\textsuperscript{40} For example, see Maloney and McCormick (2000), Sunde (2003), Sunde (2009), Brown (2011), Berger and Nicken (2014).
are divided into groups based on their ranking points. Namely, I introduce variable $q$ that shows which quantile of the points’ empirical distribution a contestant belongs to. This classification is important because various groups may attach different valuations to the points dimension.\footnote{One additional point can be of higher importance for players with lower ranks.}

**Definition 3.** Let $D$ be a strictly ordered set of threshold values ($d$) with at least two elements ($\#D \geq 2$). Player $i$ belongs to group $k$ ($q_i = k$) if and only if

$$\text{points}_i \in [d_{#D-k}, d_{#D-k+1})$$

where $\text{points}_i$ denotes a number of ranking points player $i$ holds, $k = \{1, \ldots, \#D - 1\}$.\footnote{In the structural model I use the following specification of $D$: $D = \{0, q(.15), q(.4), q(.6), q(.75), q(.85), q(.92), q(.97), q(.99), q(1)\}$}

Other relevant variables include:\footnote{Summary statistics can be found in Table 8.}

1. Tournament-specific controls: dummies for years 2009–2015, fixed effects of the two contests, reward schedules, and the total prize money.

2. Players’ characteristics: age, the body mass index ($BMI$), home bias, a number of ranking points, and whether a contestant was seeded or not.

3. Match-specific features: approximated heterogeneity between players, a binary variable that indicates if a game took place on Day 1 or Day 2.\footnote{All first-round matches are scheduled for first two days of Grand Slam tournament.}

Finally, I calculate the following empirical moments and use them to evaluate the predicting power of the structural model:

- Average winning probabilities for two contestant types (i.e. $g$ and $n$) and seeded players;

- Average winning probabilities for different age and ranking groups.

In the theoretical model, I defined the “greediest-to-win” ($g$) and not the “greediest-to-win” ($n$) types based on players’ skill-valuation profiles. Obviously, this cannot be directly observed in the data. Nevertheless, I propose two ways to approximate contestants’ identities:
1. Player $i$ is treated as the “greediest-to-win” in match $j$ ($g_j = i$) if he holds more ranking points than the competitor:

$$g_j = i \iff \text{points}_{i,j} > \text{points}_{-i,j}$$

The ATP rankings include all tournaments where players performed in last 52 weeks. Thus, the previous success must map into stronger preferences over winning and / or better skills.

2. Player $i$ is treated as the “greediest-to-win” in match $j$ ($g_j = i$) if his winning probability derived from betting odds ($p^B_{i,j}$) exceeds the one of the opponent:

$$g_j = i \iff p^B_{i,j} > p^B_{-i,j}$$

Betting odds can aggregate more information than just ranking points. As a consequence, this approach may identify contestants’ types better than the previous one.

The partition based on contestants’ age looks as follows:

$$d_{k}^{age} \in D^{age}, D^{age} = \{\min_i (age_i) , 20, 25, 30, \max_i (age_i) \}$$

$$i \in k \iff age_i \in [d_{k}^{age}, d_{k+1}^{age}) , k = \{1, ..., 4\}$$

where $k$ is the group. Variable $q$ defines the partition driven by ranking points.

4.2 Structural Estimation

4.2.1 The Model Setup

To recover contestants’ skill-valuation profiles, I explore the variation in monetary prize schedules across tournaments / years, observed effort choices, and player-specific controls. The proposed theoretical model delivers clear predictions that can be validated empirically. I formulate three results that map underlying skills and preferences into realized effort and match outcomes given the observed variation in a prize scheme.\(^{45}\) Suppose the winning benefit in dimension $A$ grows at the cost of the losing reward, i.e. the corresponding prize spread ($\triangle_A$) gets larger.

**Result 1.** For any match $j$, a skill-valuation profile is such that expected aggregate effort decreases in $\triangle_A$ (given $\triangle_B$) if and only if

1. The expected effort of player $g$ ($n$) increases (decreases) in $\triangle_A$, and the latter effect dominates:

$$\frac{\partial E_g (\cdot)}{\partial \triangle_A} > 0, \quad \frac{\partial E_n (\cdot)}{\partial \triangle_A} < 0 \quad \text{and} \quad \left| \frac{\partial E_g (\cdot)}{\partial \triangle_A} \right| < \left| \frac{\partial E_n (\cdot)}{\partial \triangle_A} \right|$$

\(^{45}\)See Section 3 for technical details.
2. The expected probability that player $g$ ($n$) wins increases (decreases) in $\Delta_A$:

$$\frac{\partial P_{\text{win}}^g (\cdot)}{\partial \Delta_A} > 0, \quad \frac{\partial P_{\text{win}}^n (\cdot)}{\partial \Delta_A} < 0$$

**Result 2.** For any match $j$, a skill-valuation profile is such that expected aggregate effort increases in $\Delta_A$ (given $\Delta_B$) if and only if

1. It is either the expected effort of both players increases in $\Delta_A$

$$\frac{\partial E_i (\cdot)}{\partial \Delta_A} > 0, \quad i = \{g, n\}$$

or the expected effort of player $g$ ($n$) increases (decreases) in $\Delta_A$, and the former effect dominates:

$$\frac{\partial E_g (\cdot)}{\partial \Delta_A} > 0, \quad \frac{\partial E_n (\cdot)}{\partial \Delta_A} < 0 \quad \text{and} \quad \left| \frac{\partial E_g (\cdot)}{\partial \Delta_A} \right| > \left| \frac{\partial E_n (\cdot)}{\partial \Delta_A} \right|$$

2. The expected probability that player $g$ wins either (weakly) increases or decreases in $\Delta_A$:

$$\frac{\partial P_{\text{win}}^g (\cdot)}{\partial \Delta_A} \geq 0 \iff \frac{\alpha_g}{\beta_g} \geq \frac{\alpha_n}{\beta_n} \quad \text{or} \quad \frac{\partial P_{\text{win}}^g (\cdot)}{\partial \Delta_A} < 0 \iff \frac{\alpha_g}{\beta_g} < \frac{\alpha_n}{\beta_n}$$

**Result 3.** For any match $j$, a skill-valuation profile is such that contestants are not sensitive to incentives in dimension $B$ if and only if$^{46}$

1. The expected effort of both players increases in $\Delta_A$:

$$\frac{\partial E_i (\cdot)}{\partial \Delta_A} > 0, \quad i = \{g, n\}$$

2. The expected probability that player $g$ wins stays constant when $\Delta_A$ grows:

$$\frac{\partial P_{\text{win}}^g (\cdot)}{\partial \Delta_A} = 0$$

The proposed identification strategy employs these theoretical results, the variation in prize schedules, approximated contestants’ effort / types, and the information about winning players to restore different skill-valuation profiles (other identification assumptions will be discussed later).

Let money and ranking points (career concerns) be labeled as goods $A$ and $B$, respectively. To bring the theoretical framework to the data, I introduce a setup similar to a random utility

---

$^{46}$The opposite holds when players do not respond to incentives in dimension $A$. 

model.\textsuperscript{47} Suppose contestants get additional individual-specific utility (or disutility) when lose:\textsuperscript{48}
\[
U_i^L (A^L, B^L, \varepsilon_i) = c_i \left( \alpha_i A^L + \beta_i B^L \right) + \varepsilon_i, \varepsilon_i \sim F_i(\varepsilon), \varepsilon \in [\varepsilon_i, \bar{\varepsilon}_i]
\]
where \(F_i(\varepsilon)\) is a truncation of a mean zero normal distribution with a standard deviation \(\sigma_\varepsilon\):\textsuperscript{49}
\[
f_i(\varepsilon) = \frac{\frac{1}{\sigma_\varepsilon} \phi \left( \frac{\varepsilon}{\sigma_\varepsilon} \right)}{\Phi \left( \frac{\varepsilon}{\sigma_\varepsilon} \right) - \Phi \left( \frac{\bar{\varepsilon}_i}{\sigma_\varepsilon} \right)}, \varepsilon \in [\varepsilon_i, \bar{\varepsilon}_i]
\]
I assume the shocks are independently drawn from the underlying player-specific distributions. Suppose both competitors, but not an econometrician, observe \(\varepsilon_i\).\textsuperscript{50} All Grand Slams match players at random in first-round games. Then, one can treat idiosyncratic \(\varepsilon_i\) as the utility (or disutility) of losing against a particular opponent.\textsuperscript{51} With this approach, contestants’ types become random:
\[
\tilde{t}_i = c_i \alpha_i \Delta_A + c_i \beta_i \Delta_B - \varepsilon_i = t_i - \varepsilon_i
\]
where \(\Delta_A\) and \(\Delta_B\) specify the difference between winning and losing prizes in corresponding dimensions. In particular, \(\tilde{t}_i\) is drawn from a truncated normal distribution (a linear transformation of \(F_i(\varepsilon)\)) and has the following characteristics: \textsuperscript{52}
\[
E\left(\tilde{t}_i\right) = t_i - E(\varepsilon_i), \text{Var}\left(\tilde{t}_i\right) = \text{Var}\left(t_i - \varepsilon_i\right)
\]
Player \(i\) is the “greediest-to-win” (\(\tilde{t}_i = \tilde{t}_g\)) if and only if \(\tilde{t}_i > \tilde{t}_{-i}\):
\[
\tilde{t}_i > \tilde{t}_{-i} \iff \varepsilon_i - \varepsilon_{-i} < t_i - t_{-i}
\]
The proposed theoretical model restricts the support of contestants’ strategies (see Section 3). Specifically, their effort levels cannot exceed \(\min\{\tilde{t}_i, \tilde{t}_{-i}\}\):
\[
e_i \leq \min\{\tilde{t}_i, \tilde{t}_{-i}\} = \min\{t_i - \varepsilon_i, t_{-i} - \varepsilon_{-i}\} \equiv \tilde{t}_n
\]
\textsuperscript{47}One can find similar settings in applied studies of auctions or industrial organization. Bajari et al. (2010) provide the profound overview of modern approaches one can use to recover underlying parameters of static and dynamic games with different information structures. Authors indicate that often structural inference involves conditioning on reduced forms. As a result, the estimation becomes equivalent to a single-agent problem. Donald and Paarsch (1996), Bajari and Hortacsu (2003) work with the auction setting and formulate a parametric likelihood estimator based on players’ winning probabilities. Degan (2007), Degan and Merlo (2011) provide coherent examples of the likelihood construction in the individual choice setting where player-specific characteristics (age, education etc.) matter.
\textsuperscript{48}The additive separable noise must be introduced in an asymmetric way. If it has the same effect on winning and losing utility, \(\varepsilon_i\) cancels out when one defines contestants’ types (\(t_i\)). Since winning probabilities used to construct the likelihood function depend on \(t_i\), the symmetric noise setting becomes deterministic.
\textsuperscript{49}See Appendix C for the discussion of the player-specific noise distribution.
\textsuperscript{50}I justify non-observability of \(\varepsilon_i\) for the econometrician as follows. Contestants have much more information about idiosyncrasies affecting their opponents (training regime, health- and career-related concerns, other specific circumstances etc.). Moreover, they can see each other’s play in different competitions and draw conclusions about the game style and tactical steps a particular individual uses. All these things may shape contestants’ perception of losing. However, they are hard to observe and aggregate for the econometrician.
\textsuperscript{51}For example, take the contestant ranked 100 who can lose against either one of top-5 athletes or a player with similar characteristics. In the former case, he may feel less discouraged and even perceive this event as a valuable experience.
\textsuperscript{52}As Appendix C shows, \(E\left(\tilde{t}_i\right)\) and \(\text{Var}\left(\tilde{t}_i\right)\) depend on player- and competitor-specific controls.
The latter inequality defines a player-specific upper bound on $\varepsilon_i$ ($\bar{\varepsilon}_i$) that depends on the parameters of interest (namely, contestants' skills and preferences). Appendix C specifies other conditions the distribution of $\varepsilon_i$ must satisfy to match the theoretical framework. Thus, the econometric model is characterized by parameter-dependent support. This feature has important implications for the estimation procedure, and I discuss them later.\textsuperscript{53}

Let $\pi$ and $x$ denote the sets of parameters and controls, respectively. Given that players' skills ($c$) interact with their preferences ($\alpha$ and $\beta$) in a multiplicative way, one cannot identify these values separately without imposing additional restrictions. Define a joint effect of $c$ and $\alpha$ ($\beta$) as $\tilde{\alpha}$ ($\tilde{\beta}$) and express it as a function of $\pi$ and $x$:

$$\tilde{\alpha}(\pi, x) \equiv c(\pi, x, \alpha)$$
$$\tilde{\beta}(\pi, x) \equiv c(\pi, x, \beta)$$

where $\pi$ and $x$ reflect subsets of $\pi$ and $x$, respectively, that shape a value of $l$, $l = \{c, \alpha, \beta, \tilde{\alpha}, \tilde{\beta}\}$. Estimating $\tilde{\alpha} (\cdot)$ and $\tilde{\beta} (\cdot)$ is sufficient to capture the relative importance of different prize dimensions for player $i$ (i.e. $\frac{\tilde{\alpha}(\pi, x)}{\tilde{\beta}(\pi, x)}$) and measure the degree of heterogeneity between the competitors.

To identify $\tilde{\alpha}$ and $\tilde{\beta}$, I exploit the variation in player- and tournament-specific characteristics. The main challenge is to find the instruments to isolate the effects of $\tilde{\alpha}$ and $\tilde{\beta}$. I impose the following functional assumptions: \textsuperscript{54}

$$c(\pi, x) = \exp(\pi_1^{\text{points}}_i + \pi_2^{\text{age}}_i + \pi_3^{\text{age}}_2 + \pi_4^{\text{BMI}}_i + \pi_5^{\text{bias}}_i + \pi_6^{\text{Tour}} + \pi_7^{Z})$$

$$\tilde{\alpha}(\pi, x) = c(\pi, x) \exp(\pi_1^{\tilde{\alpha}} + \pi_2^{\tilde{\alpha}})$$

$$\tilde{\beta}(\pi, x) = c(\pi, x) \exp(\pi_1^{\tilde{\beta}} + \pi_2^{\tilde{\beta}})$$

where:

- $\text{points}_i$ is a number of ranking points contestant $i$ holds;
- $\text{BMI}_i$ reflects the body mass index;
- $\text{bias}_i = 1$ if $i$ plays in his home country and $\text{bias}_i = 0$, otherwise;
- $\text{Tour} = 1$ for the Australian Open and $\text{Tour} = 0$ for the French Open;
- $Z$ represents a set of dummy variables that includes:

\textsuperscript{53}For example, see Hirano and Porter (2003).
\textsuperscript{54}The exponents guarantees non-negativeness of skills and preference parameters.
- A day when match \( j \) took place (\( d_j = 1 \) if match \( j \) was scheduled for Day 1; otherwise, \( d_j = 0 \));\(^{55}\)
- A year-specific effect (\( t \));
- Interaction terms \( Tour \times d_j \) and \( d_j \times t \).\(^{56}\)

- \( seed_i = 1 \) if \( i \) was seeded in the tournament draw and \( seed_i = 0 \), otherwise;\(^{57}\)
- \( q_i \) reflects to which quantile of the points’ empirical distribution player \( i \) belongs to (see Subsection 4.1).

Obviously, a number of ranking points (\( points_i \)) conveys the information about players’ skills. Intuitively, top-athletes have more experience and display better game statistics than their lower-ranked peers. Generally, players perform worse in the beginning and closer to the end of their career. To capture this \( U \)-shaped pattern, I include \( age_i \) and \( age_i^2 \) into the \( c(\pi_c, x_i^c) \) specification. Since tennis is a physically intensive game, athletes with bigger \( BMI_i \) values may have an advantage in serve speed and win more often. On the other hand, higher \( BMI_i \) can result in slower running. Thus, the effect of \( BMI_i \) on athletes’ performance is not ex-ante clear. Contestants playing in their home countries may know the courts better, feel more support, avoid acclimatization. These factors will work in their favor and improve the skills. Variables \( Tour \) and \( Z_{jt} \) are included to account for contest- and time-specific shocks all players face (a type of the surface, geographical location, weather conditions etc.).

It is assumed that individual preferences over prize items (namely, \( \tilde{\alpha}(\cdot) \) and \( \tilde{\beta}(\cdot) \)) are imperfectly correlated with contestants’ rankings. Top–32 seeded players (\( seed_i = 1 \)) have already advanced in their career and earned a significant amount of prize money. Consequently, their preferences over this good can differ from those lower-ranked competitors display.\(^{58}\) Hence, \( seed_i \) must affect \( \tilde{\alpha}(\cdot) \). Next, \( \tilde{\beta}(\cdot) \) (a valuation attached to a non-monetary dimension) has to depend on contestant \( i \)’s position in the points’ empirical distribution captured by the \( q_i \) variable. For example, one additional point can be more important for players with lower ranks because it grants them access to better contests and improves career perspectives.\(^{59}\) The observed variation in monetary prize schedules across tournaments and years, coupled with fixed rewards in the non-monetary dimension, also helps in isolating the effect of \( \tilde{\alpha}(\cdot) \) from \( \tilde{\beta}(\cdot) \).

\(^{55}\) All first-round matches take place on first two days of the tournaments.

\(^{56}\) I do not include \( Tour \times t \) into \( Z_{jt} \) because it perfectly correlates with contest-specific monetary prize schedules.

\(^{57}\) Every Grand Slam has 32 seeded contestants (25% of all players) who never meet each other in the first round. These are top-ranked athletes with enough experience and a high probability of being advanced in the tournament.

\(^{58}\) For example, a marginal value of money can be higher for non-seeded players who, in case of winning the prize, get a chance to invest in their training and perform better in the future.

\(^{59}\) On the contrary, top-players may be more career-concerned, which corresponds to higher values of \( \beta \)’s.
The way Grand Slams match players and the modeling assumptions imposed on $\hat{\alpha}(\cdot)$ and $\hat{\beta}(\cdot)$ make it possible to identify the $\sigma_e$ parameter (the variance of the underlying noise distribution). First, the fact that players are randomly paired excludes any kind of selection bias driven by the strategic choice of an opponent. Second, adding the $\text{Tournament}$ variable and a set of time- and tournament-specific dummies, together with their interaction terms ($Z_{jt}$), allows me to isolate common shocks contestants face. Hence, all the variance left can be attributed to a random noise term.

I characterize first-round winning prizes as a continuation value of being advanced in the contest. Suppose success and failure at later stages of the tournament are equally probable. Also, assume that identities of potential future opponents do not matter. With this approach, prize spreads are equal for all players. As a result, heterogeneity in first-round games stems only from different skills and asymmetric preferences.

To estimate the model, I treat matches as independent and formulate the likelihood in terms of players’ winning and losing probabilities. Let $P_{gj}$ ($P_{nj}$) denote a probability that type $g$ ($n$) wins match $j$. Since player $n$ stays inactive with probability $p_n^0 \in (0, 1)$, $P_{ij}$ must include two components – $P_{ij}^1$ and $P_{ij}^2$ (see Proposition 1):

1. With probability $p_n^0$, type $g$ certainly wins:
   $$P_{gj}^*(\pi, \varepsilon_{gj}, \varepsilon_{nj}) = p_n^0, \; P_{nj}^*(\pi, \varepsilon_{gj}, \varepsilon_{nj}) = 0$$

2. Otherwise, type $g$ ($n$) wins with probability $\left\{ \frac{e_{gj}}{t_{nj}} \right\} \left\{ \frac{e_{nj}}{t_{nj}} \right\}$:
   $$P_{gj}^{**}(\pi, \varepsilon_{gj}, \varepsilon_{nj}, e_{gj}) = (1 - p_n^0) \frac{e_{gj}}{t_{nj}}, \; P_{nj}^{**}(\pi, \varepsilon_{gj}, \varepsilon_{nj}, e_{nj}) = (1 - p_n^0) \frac{e_{nj}}{t_{nj}}$$
   where $e_{gj}^0$ ($e_{nj}^0$) corresponds to observed effort exerted by type $g$ ($n$) in match $j$.

Then, contestant $i$’s winning probability can be calculated as a sum of the two components, $P_{ij}^*(\cdot)$ and $P_{ij}^{**}(\cdot)$:
$$P_{ij}(\pi, \varepsilon_{gj}, \varepsilon_{nj}, e_j^i) = P_{ij}^*(\pi, \varepsilon_{gj}, \varepsilon_{nj}) + P_{ij}^{**}(\pi, \varepsilon_{gj}, \varepsilon_{nj}, e_j^i)$$

and a complete data likelihood for match $j$ looks as follows:
$$L_j^c(\pi \mid W_{1j}, e_{1j}, e_{2j}, x_1, x_2, \varepsilon_{1j}, \varepsilon_{2j}) = I \{1 = g\} \left[ (\left[ P_{gj}^1 \cdot (1 - P_{nj}^2) \right] + I\{W_{1j} = 1\} \right] \left[ (1 - P_{gj}^1) \cdot P_{nj}^2 \right]^{1 - I\{W_{1j} = 1\}} + (1 - I \{1 = g\}) \left[ (\left[ P_{nj}^1 \cdot (1 - P_{gj}^2) \right] + I\{W_{1j} = 1\} \right] \left[ (1 - P_{nj}^1) \cdot P_{gj}^2 \right]^{1 - I\{W_{1j} = 1\}} \right]$$

60. This strategy was used in several empirical papers (for instance, see Silverman and Seidel (2011), Ivankovic (2007)) with the following argument: winning and losing probabilities of heterogeneous players average to “$1/2 - 1/2$”.
61. This formulation is similar to a standard probit / logit model.
where

- \( P_{gj}^l \equiv P_{gj} (\pi, \varepsilon_{1j}, \varepsilon_{2j}, e_j^l) \) and \( P_{nj}^l \equiv P_{nj} (\pi, \varepsilon_{1j}, \varepsilon_{2j}, e_j^l) \) for \( l = \{1, 2\} \);
- \( W_{1j} = 1 \) if player 1 wins match \( j \);
- \( I \{1 = g\} = 1 \) if player 1 is of type \( g \);
- \( x_1 \) and \( (x_2) \) reflect individual characteristics of players 1 and 2, respectively.

With this likelihood specification, I use the information on both realized effort levels, \( e_{1j} \) and \( e_{2j} \), that are generally not equal.\(^{62}\) Exploiting the variation in \( e_{1j} \) and \( e_{2j} \) helps me to recover supports of the contestants’ strategies, which depend on \( \tilde{t}_{nj} \), and identify the parameters shaping \( \tilde{\alpha}(x) \) and \( \tilde{\beta}(x) \).

Since \( \varepsilon_{1j} \) and \( \varepsilon_{2j} \) are unobservable, one must calculate a probability that contestant 1 is the “greediest-to-win” in match \( j \):

\[
P_{1j}^g (\pi, x_1, x_2) \equiv P (1 = g) = \int_{u_j}^{t_1 - t_2} f_j (u) \, du, \quad u_j = \varepsilon_{1j} - \varepsilon_{2j}
\]

where \( u_j = \varepsilon_{1j} - \varepsilon_{2j} \) and \( t_i = \tilde{\alpha}(x_i) \Delta_A + \tilde{\beta}(x_i) \Delta_B, \quad i = \{1, 2\} \). Replacing \( I \{1 = g\} \) with \( P_{1j}^g (\pi, x_1, x_2) \) and taking conditional expectations over \( \varepsilon_{1j} \) and \( \varepsilon_{2j} \), I formulate an expected likelihood:

\[
L_j^e (\pi \mid W_{1j}, e_{1j}, e_{2j}, x_1, x_2) = \frac{P_{1j}^g (\pi, x_1, x_2) E_{\varepsilon_{nj}, e_{nj}} \left[ (P_{gj}^1 \cdot (1 - P_{nj}^2))^{W_{1j} = 1} \cdot (1 - P_{nj}^1) \cdot \left( P_{nj}^2 \right)^{1 - I(W_{1j} = 1)} \mid 1 = g \right]}{1 - P_{1j}^g (\pi, x_1, x_2)} E_{\varepsilon_{nj}, e_{nj}} \left[ (P_{nj}^1) \cdot (1 - P_{gj}^2)^{I(W_{1j} = 1)} \cdot (1 - P_{nj}^1) \cdot \left( P_{gj}^2 \right)^{1 - I(W_{1j} = 1)} \mid 1 = n \right]
\]

When one takes a logarithm of \( L_j^e (\pi \mid W_{1j}, e_{1j}, e_{2j}, x_1, x_2) \) and aggregates over \( K \) matches, the objective function to maximize becomes:

\[
\log \sum_{j=1}^K \log \left[ L_j^e (\pi \mid W_{1j}, e_{1j}, e_{2j}, x_1, x_2) \right]
\]

Finally, I can write down a complete optimization program with a single constraint:

\[
\max_x \left\{ \log L^e (\pi \mid W, e, x) \right\}
\]

\[
s.t. \max \{e_{1j}, e_{2j}\} \leq \min \{U_{gj}^W, U_{nj}^W\} \quad \forall j = \{1...K\}
\]

where the inequality guarantees well-defined supports for player-specific noise distributions (see Appendix C). Since the constructed econometric model features parameter-dependent support, it

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\(^{62}\)If \( e_j^l = e_j^{-l} \), it must be \( P_{gj} (\cdot) = (1 - P_{nj} (\cdot)) \). The same holds if one focuses on expected winning probabilities and does not use the information on realized effort for both players.
violates a standard regularity condition of the maximum likelihood estimation. Specifically, the solution of the optimization program does not need to be interior. On top of this, the estimator is not necessarily asymptotically normal. To address these issues, I develop the following approach. First, a derivative-free numerical optimization routine that allows for corner solutions is employed to solve the program. Second, to avoid any results that rely on the assumption of asymptotic normality, I compute bootstrap standard errors. To study the properties of the estimator in more detail, I run simulations.\footnote{Simulation results are available by request} Despite the indicated non-regularity, the estimator is consistent. However, as Hirano and Porter (2003) point out, it can be asymptotically inefficient and must be improved later on.\footnote{The lack of asymptotic efficiency is the least problematic property that generally results in bigger standard errors.}

4.2.2 Results and the Goodness–of–Fit

In the beginning, I follow the baseline and estimate the distribution of contestants’ types for the two-dimensional model (see Table 1, column 1). Most of the variables are statistically significant. As expected, more ranking points map into better skills. Importantly, non-monetary incentives matter for effort provision: the respective regression coefficients, $\pi_1^3$ and $\pi_2^3$, are statistically significant (see Table 1, column 1). Higher ranks affect contestants’ preferences over two prize dimensions in the following way. Being seeded $(seed_i = 1)$ decreases the marginal utility of money. At the same time, top-players (those with lower values of $q_i$) tend to value the non-monetary dimension more and display stronger career concerns.

The presence of home bias has a positive effect on athletes’ performance. This finding is in line with the previous empirical tests of the contest theory. On average, individuals play better when the body mass index declines. Skills indeed turn to be non-linear in contestants’ age: both $age_i$ and $age_i^2$ are statistically significant (see Table 1, column 1). Given that the coefficient in front of $age_i^2$ is negative, one can approximate when the skills achieve their maximum over $age$ (keeping other player-specific characteristics constant):

$$age^* = \arg \max_{age_i} \left[ \ln \left\{ c \left( \pi_c, x_{cjt}^{ij} \right) \right\} \right] \approx 27.5$$

Finally, contestants tend to perform better in the Australian Open. This effect can be explained as follows. The French open is played on clay courts, while the Australian Open uses hard ones. On average, the athletes find it easier to compete on the latter surface. As a result, one can expect stronger skills in the Australian Open.

Estimated heterogeneity is summarized in Table 1.9. On average, the Australian Open features more uneven matches than the French Open. However, this pattern can be driven by the aforementioned contest-specific effect. For example, take a match from the French Open with
heterogeneity \( H^{FO} \). If the same couple plays in the Australian Open, \( H^{FO} \) must be multiplied by \( e^{\pi \theta} \approx 1.08 \), and \( H^{AO} > H^{FO} \) (see Table 1 for the estimates). Thus, *ceteris paribus*, heterogeneity is stronger in the latter competition.

Table 1: Structural Estimation: Results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Both Goods</th>
<th>Only A</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_{1}^{\beta} )</td>
<td>-25.709*** (3.939)</td>
<td>.662*** (.157)</td>
</tr>
<tr>
<td>( seed_{i} )</td>
<td>-18.899*** (2.311)</td>
<td>-.483*** (.158)</td>
</tr>
<tr>
<td>( \pi_{1}^{\beta} )</td>
<td>.784*** (.152)</td>
<td>-</td>
</tr>
<tr>
<td>( q_{i} )</td>
<td>-2.895*** (.079)</td>
<td>-</td>
</tr>
<tr>
<td>( points_{i} )</td>
<td>1.994** (.906)</td>
<td>6.988*** (1.528)</td>
</tr>
<tr>
<td>( age_{i} )</td>
<td>4.473*** (.219)</td>
<td>-3.345*** (.136)</td>
</tr>
<tr>
<td>( age_{i}^{2} )</td>
<td>-4.502*** (.851)</td>
<td>2.007*** (.328)</td>
</tr>
<tr>
<td>( BMI_{i} )</td>
<td>-.389** (.155)</td>
<td>-.837*** (.132)</td>
</tr>
<tr>
<td>( hbias_{i} )</td>
<td>.112** (.045)</td>
<td>.126 (.203)</td>
</tr>
<tr>
<td>( Tour )</td>
<td>.074 (.051)</td>
<td>.087 (.194)</td>
</tr>
<tr>
<td>( \sigma_{e} )</td>
<td>213.078*** (3.632)</td>
<td>195.143*** (6.498)</td>
</tr>
</tbody>
</table>

Time- and Tournament-Specific Controls

| logL | -969.27 | -1343.54 |
| LR-test | \( LR = 748.6 > \chi^{2}(2) \) |

Note: individual effort is measured as \( \tilde{e}_{ij} = \tilde{u} - \tilde{u}_{ij} \) where \( \tilde{u}_{ij} \) is a number of unforced errors per set played, \( \tilde{u} = \max_{i,j} \tilde{u}_{ij} \). Variables \( points_{i} \), \( BMI_{i} \), \( age_{i} \) are rescaled:

\[
\tilde{w}_{i} = \frac{w_{i} - \min(w)}{\max(w) - \min(w)}
\]

where \( w_{i} (\tilde{w}_{i}) \) denotes the original (rescaled) value of the variable for player \( i \), \( w \) is a vector of \( w_{i} \). Time- and tournament-specific controls correspond to \( Z_{ij} \) specified in Subsection 4.2.1. \( K \) denotes a number of matches. Bootstrap standard errors are reported in brackets. Values with *, **, and *** correspond to 90%, 95%, and 99% significance.
As an attempt to isolate this channel, I re-calculate heterogeneity assuming no contest-specific effect (see Table 1.10). In this case, matches become more uneven in the French Open. Additionally, a range of estimated heterogeneity in this competition is bigger than in the Australian Open (the pattern also holds with the contest-specific effect). One could explain these observations with matching policies used in the two tournaments. Ideally, players must be paired randomly. However, this does not seem to be the case in some Grand Slams. For instance, in the US Open, top-players systematically get weaker opponents than predicted by random matching.\textsuperscript{65} As a result, average heterogeneity may increase. If the French Open (but not the Australian Open) adopts similar practices, this might explain the pattern discovered in the data.

Next, the model is simulated 1'000 times. To evaluate the goodness-of-fit, I calculate moments specified in Subsection 4.1 and match them against empirical counterparts. In the beginning, consider simulated and actual winning probabilities for two contestant types (namely, \(g\) and \(n\)) and seeded players (see Table 2). Overall, the model replicates type-specific winning probabilities well. It is successful in explaining the difference in contestants’ behavior and captures the gap between \(P^W_g\) and \(P^W_n\). Also, the model accounts for the fact that seeded players win more often than non-seeded \(g\)-types (\(P^W > P^W_g\)).

Table 2: Simulated vs. Actual Type-Specific Winning Probabilities: Two-Dimensional Model

<table>
<thead>
<tr>
<th>The Model</th>
<th>The Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points–Based Classification</td>
<td>Bets–Based Classification</td>
</tr>
<tr>
<td>((P^W_g, P^W_n))</td>
<td>(.721, .279)</td>
</tr>
<tr>
<td>Seed–Based Classification</td>
<td>(.726, .276)</td>
</tr>
<tr>
<td>(P^W)</td>
<td>(.763, .244)</td>
</tr>
<tr>
<td>.824</td>
<td>.844</td>
</tr>
</tbody>
</table>

Further, I implement the division based on contestants’ age and ranking points (see Subsection 4.1). Figure 4 plots simulated and actual winning probabilities for all relevant groups. The age-driven classification manages to match empirical patterns: the two curves are very close to each other. The model predicts the highest winning probability for contestants between 25 and 30. This pattern is in line with the data. The model also replicates an empirical curve under the points-based classification. Actual and predicted winning probabilities are very close in upper quantiles of the points’ distribution. The model shows a slight tendency towards overestimating (underestimating) winning probabilities for lower-ranked (middle-ranked) contestants. However, these patterns offset each other.

\textsuperscript{65}For example, see http://www.espn.com/espn/otl/story/_/id/6850893/espn-analysis-finds-top-seeds-tennis-us-open-had-easier-draw-statistically-likely.
In Appendix D, I also check if the structural setup replicates stylized facts observed in players’ effort. As before, contestants are divided into age- and points-based groups. In addition, I split the matches by the degree of heterogeneity between competitors. Overall, the model is successful in reproducing shapes of the respective empirical curves (see Figure 12 and Appendix D for the discussion).

To evaluate the consequences of neglecting one good, I also estimate a specification with monetary prizes only, as always done in the empirical contest literature (see Table 1, column 2). The two-dimensional model shows the highest log-likelihood. On top of this, the LR-test rejects a null hypothesis that the baseline specification is equivalent to the single-item alternative.\footnote{The testing statistic looks as follows:}

$$LR = 2 \left( \log L_{\text{complete}} - \log L_{\text{restricted}} \right)$$

where $\log L_{\text{complete}}$ ($\log L_{\text{restricted}}$) is taken from the two-dimensional model (single-item specification). Given that both restricted models have two parameters less, the statistic must be distributed as $\chi^2 (2)$.
Table 3: Simulated vs. Actual Type-Specific Winning Probabilities: Only Monetary Prizes

<table>
<thead>
<tr>
<th>The Model</th>
<th>The Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points–Based</td>
<td>Bets–Based</td>
</tr>
<tr>
<td>Classification</td>
<td>Classification</td>
</tr>
<tr>
<td>((P_n^W, P_g^W))</td>
<td>(.608, .392) (.726, .276)</td>
</tr>
<tr>
<td>Seed–Based Classification</td>
<td>(.763, .244)</td>
</tr>
<tr>
<td>(P^W)</td>
<td>.644 \quad .844 \quad .844</td>
</tr>
</tbody>
</table>

and the corresponding degree of heterogeneity \((H_\tilde{\alpha})\) for one- and two-dimensional models (see Table 4). First, the single-item setting underestimates heterogeneity in the monetary dimension. This fact explains why the setup generates type-specific winning probabilities that are closer to “1/2–1/2” than observed. Second, it predicts higher valuations attached to money and, consequently, overestimates the incentive effect of this prize. Thus, if one reward component is disregarded, the estimates become biased, and this may lead to incorrect policy recommendations.

Overall, the structural analysis shows that contestants, indeed, respond to multi-dimensional incentives. This finding reveals the directions to improve the existing reduced-form tests of the contest theory that always deal with single-item prizes.

Table 4: One vs. Two Dimensions: Estimated Heterogeneity

<table>
<thead>
<tr>
<th>Two-Dimensional Model</th>
<th>One-Dimensional Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean predicted valuations in dimension (A)</td>
<td>.12 \quad 1.84</td>
</tr>
<tr>
<td>Ratio of mean predicted (H_\tilde{\alpha}) (rescaled): (\frac{H_\tilde{\alpha}^2}{H_\tilde{\alpha}^1})</td>
<td>20.7</td>
</tr>
<tr>
<td>Ratio of predicted (Var(H_\tilde{\alpha})) (rescaled): (\frac{Var(H_\tilde{\alpha}^2)}{Var(H_\tilde{\alpha}^1)})</td>
<td>6.51</td>
</tr>
</tbody>
</table>

Note: \(H_{\tilde{\alpha}, j}^1 = |\hat{\alpha}^1(x_{ij}) - \hat{\alpha}^1(x_{-ij})|\) \((H_{\tilde{\alpha}, j}^2 = |\hat{\alpha}^2(x_{ij}) - \hat{\alpha}^2(x_{-ij})|)\) is heterogeneity between contestants in match \(j\) in the single-item (two-dimensional) model; to calculate comparable means and standard deviations for the two series, \(H_\tilde{\alpha}\) is rescaled as follows:

\[
h_\tilde{\alpha}, j = \frac{H_{\tilde{\alpha}, j} - \min(H_{\tilde{\alpha}})}{\max(H_{\tilde{\alpha}}) - \min(H_{\tilde{\alpha}})}
\]

where \(H_{\tilde{\alpha}}\) is a vector of \(H_{\tilde{\alpha}, j}\).
4.2.3 Policy Experiments

In this section, I ask if there exist alternative prize schemes that induce more aggregate effort for estimated skill-valuation profiles and the given budget of the contest organizer. In principle, one can abstract from the tennis-specific setting and think about the job promotion context. As it was discussed earlier, ranking points map into career concerns. Thus, the proposed policy experiments have a sufficient degree of generality.

As before, I assume that the designer seeks to maximize expected aggregate effort.\(^{67}\) In every match \(j\), he can allocate endowments of money and ranking points (\(A\) and \(B\), respectively) between winning and losing bundles. Since winners obtain a continuation value of being advanced in the contest, I assume that losing prizes (\(A_L\) and \(B_L\)) increase (decrease) at cost (benefit) of the final trophy. Then, one can restore the “Winner–Takes–All” schedule as follows:

\[
\hat{A} \equiv A_0^W = A_a^W + p_{final}A_a^L \\
\hat{B} \equiv B_0^W = B_a^W + p_{final}B_a^L
\]

where:

- \((A_a^W, B_a^W)\) and \((A_a^L, B_a^L)\) correspond to actual winning and losing prizes;
- \(p_{final}\) is a probability to get to the final.\(^{68}\)

To run the experiments, I construct grids \(\bar{A}^L\) and \(\bar{B}^L\) including actual losing prizes:

\[
\bar{A}^L = [0, \bar{u}A_a^L], \quad \bar{B}^L = [0, \bar{u}B_a^L], \quad \bar{u} > 1
\]

For every match \(j\) and prize scheme \(k\), I simulate the model 1,000 times, calculate expected effort and take the median. At the contest level, a mean over all games is taken.

First, assume **prizes in dimension \(B\) (career concerns) are fixed**, i.e. the designer can shape only monetary incentives. Moreover, suppose the **reward schedules are not match–specific**. This setting mirrors the tennis example. As there exists a fixed tournament effect, I analyze the Australian Open and the French Open separately. Importantly, contest organizers never leave good \(B\) (ranking points) out because this reduces aggregate effort significantly (see Table 11). For this reason, I analyze only two-dimensional prize structures. Figures 7 and 8 display how expected effort changes when monetary losing prizes (\(A_L\)) grow in both contests. On average, the Australian Open would gain from zero losing benefits in the first round. This effect is driven by responses of relatively strong players whose expected effort

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\(^{67}\)In principle, the designer can have more objectives, especially in the dynamic contest setting. For instance, he may want relatively strong players to advance in the competition with a higher probability.

\(^{68}\)\(p_{final} = \frac{1}{2}\) under non-specific prize spreads (the “1/2–1/2” assumption).
decreases in $A^L$. However, weaker competitors behave differently. Their expected effort increases up to 2.9% (2.7%) with respect to zero (actual) losing prizes when $A^L$ grows (see Figure 7).

The French Open shows the opposite pattern. On average, the contest would benefit from higher losing prizes in first-round matches. Reallocation 5% of final winning benefits in favor of first-round losing rewards could improve mean expected aggregate effort up to .41% with respect to the actual schedule. Not surprisingly, the effect is relatively small when one must take the average over heterogeneous matches. Nevertheless, the pattern is stable and driven by responses of disadvantaged (type n) contestants (see Figure 8). In particular, these players improve their effort by more than 3.7% (3.2%) when $A^L$ grows relative to zero (actual) losing benefits. The difference between the Australian Open and the French Open can be explained by more heterogeneous matches in the latter contest (see Subsection 4.2.2 for the discussion).

Next, I investigate how players’ responses to higher losing prizes depend on the degree of heterogeneity in a particular game. All matches are divided into groups based on the absolute difference in contestants’ ranking points ($H^p_j$):^69

$$H^p_j = |points_{ij} - points_{-ij}|$$

$$d_{k, p}^{het, p} \in D^{het, p}, D^{het, p} = \{0, 200, 500, 1000, 1600, 3000, \max_j \left( H^p_j \right) \}$$

$$j \in k \Leftrightarrow H^p_j \in [d_{k, p}^{het, p}, d_{k+1}^{het, p}], k = \{1, ..., \#D^{het, p} - 1\}$$

where $k$ is the group, and $\#D^{het, p}$ denotes cardinality of $D^{het, p}$. Figures 9 and 10 depict match-specific responses to growing monetary losing prizes for the two contests. The Australian Open and the French Open show very similar patterns summarized in Table 1.5. Not surprisingly, stronger players (g types) always decrease expected effort when monetary losing prizes grow. On the contrary, the response of their opponents depends on the degree of heterogeneity. In relatively even matches ($H^p_j < 500$), higher losing benefits reduce the expected effort of disadvantaged players. If heterogeneity is strong ($H^p_j \geq 500$), the pattern changes. Increasing losing prizes improves expected effort of weaker contestants by more than 10% compared to the case of $A^L = 0$ and actual schedules (see Figure 10). This growth compensates losses caused by negative responses of advantaged players. In particular, reducing final winning prizes by 5% for the benefit of first-round losing rewards increases expected aggregate effort up to 3% in relatively heterogeneous matches (see $H^p_j \in [1000, 3000]$ in Figures 9 and 10).

All observed patterns are in line with the theoretical predictions. Under relatively weak heterogeneity, dominant players do not have a big advantage over their opponents. As a result, the designer faces no need to redistribute a relative power between competitors. Moreover,

^69I use ranking points but not estimated skill–valuation profiles to approximate heterogeneity because the former is perfectly observable. Also this measure is easier to understand and interpret for contest managers in the real-life setting.
Table 5: The Effect of Higher Monetary Losing Prizes and Contestants’ Heterogeneity

<table>
<thead>
<tr>
<th>Effect of higher $A_L$ on mean exp. agg. effort</th>
<th>$H_j^p &lt; 500$ (44% of matches)</th>
<th>$H_j^p \geq 500$ (56% of matches)</th>
</tr>
</thead>
</table>

winning benefits of both contestant types decline more or less symmetrically when higher losing prizes are introduced. In this case, the positive equilibrium effect will never compensate losses caused by the prize stake reduction. Thus, the “Winner–Takes–All” allocation induces the highest aggregate effort when players have similar skills and preferences.

If one contestant has a very strong advantage in skills or at least in one prize dimension, the opponent feels discouraged. Then, the “Winner–Takes–All” schedule does not induce enough aggregate effort. However, the designer handles this heterogeneity issue by introducing positive losing prizes. The policy makes players’ types more balanced (ideally, $t_n = t_g$), and the contest becomes homogeneous. Now, the competitor who used to be disadvantaged has higher chances to win and increases his effort. In case of pronounced heterogeneity, this positive equilibrium effect exceeds losses associated with lower winning benefits. Eventually, the total effort grows.

Next, suppose the designer can change prizes in both dimensions. Although this is not the case in the tennis application, the experiment becomes important in a more general job promotion setting. As before, the designer is constrained by fixed endowments of goods ($\hat{A}$ and $\hat{B}$) and uniform reward schedules. Figure 11 shows how mean expected aggregate effort changes with losing prizes in both contests. Overall, the Australian Open (the French Open) can improve up to .25% (.24%). Key patterns in the monetary dimension are identical to those found in the previous case. On average, the Australian Open benefits from the “Winner–Takes–All” schedule. Alternatively, higher monetary losing prizes ($A_L > A_n^L$) improve mean expected aggregate effort in the French Open. However, both contests would prefer to allocate ranking points only to first-round winners ($B^L = 0$). As a result, the aforementioned difference between the Australian Open and the French Open affects the designers’ choices only in the monetary dimension.70

Further, I use the partition introduced above and trace group-specific responses to higher losing rewards. As expected, in relatively homogeneous matches ($H_j^p < 500$) the “Winner–Takes–All” schedule provides the strongest incentives to compete. When heterogeneity increases ($H_j^p \geq 500$), positive losing prizes in both dimensions ($A_L > 0$ and $B^L > 0$) improve expected aggregate effort. In this case, the contest becomes more homogeneous, and weaker players get stimuli to compete

70In Subsection 4.2.2, I explained more heterogeneity in the French Open with the contest-specific seeding policy. Recall that variable $seed_i$ shapes only players’ valuations in the monetary dimension (see Subsection 4.2.1 for identification restrictions). Thus, experimental findings provide supportive evidence in favor of the matching-based explanation of excessive heterogeneity in the French open.
more aggressively. Tables 12 and 13 characterize group-specific reward schedules supporting the highest aggregate effort in the Australian Open and the French Open. Surprisingly, actual prize schemes \((W_a = (A^W_a, B^W_a)\) and \(L_a = (A^L_a, B^L_a)\) never induce the strongest competition. As before, relatively even pairs \((H^P_j < 500)\) perform the best when the “Winner–Takes–All” allocation is implemented. Matches that feature stronger heterogeneity \((H^P_j \geq 500)\) require more losing benefits in both dimensions to support the highest expected aggregate effort. In this case, reallocating 5% of money and 2% of ranking points from final winning prizes to first-round losing rewards could improve expected aggregate effort in the French Open (the Australian Open) by more than 4.9% (3.4%). The gain would be 5.6% (3.7%) for the French Open (the Australian Open) compared to the “Winner–Takes–All” schedule. Thus, higher losing rewards improve the contestants’ performance in heterogeneous matches.

All experiments conducted in this section can be applied to the job promotion setting directly. When managers decide how to allocate monetary and non-monetary rewards (status, other career-related stimuli etc.) between heterogeneous participants, they may want to assign positive losing prizes. However, if competing workers have similar skills and preferences, the “Winner–Takes–All” scheme induces the highest aggregate effort. In case of multiple bilateral contests with uniform prizes (for example, professional sports), the optimal allocation depends on the pool of potential competitors and the matching policy.

5 Conclusion

Job promotion, professional sports and other similar interactions can be seen as bilateral contests with heterogeneous players and rewards including multiple goods. I propose a theoretical framework to model this setting and characterize the optimal prize allocation that maximizes players’ aggregate effort. When heterogeneity in preferences and / or skills is strong, the designer must leave a positive prize for a loser. On the one hand, this schedule reduces winning benefits and, consequently, incentives to exert effort (direct effect). On another hand, higher losing prizes eliminate the advantage of a stronger player and make the contest more balanced. As a result, the previously disadvantaged opponent can win with a higher probability and gets more stimuli to compete. If heterogeneity is severe, this positive equilibrium effect dominates the direct one. Thus, expected aggregate effort increases.

If contestants’ preferences and / or skills are similar, a stronger participant has no significant advantage. In addition, higher losing rewards cause a comparable reduction in winning benefits of both players. Then, the positive equilibrium effect can never compensate losses in contestants’ effort induced by cutting the prize spread. As a result, the “Winner–Takes–All” allocation supports the highest expected aggregate effort.
Notably, positive losing rewards could never benefit the designer in a single-item case. In this setting, contestants’ types become aligned if and only if winning and losing prizes are identical. The allocation, however, gives no incentives to compete and is never optimal. On the contrary, in the two-dimensional setup it is possible to provide strictly positive winning benefits for both participants and at the same time make the contest even by increasing the losing prize. The result does not require convexity of the cost function as a traditional argument in favor of multiple positive rewards.\(^7\) The key ingredient is asymmetry in contestants’ valuations.

To highlight the importance of multi-dimensional prizes in job promotion and other similar settings, I structurally estimate the model using data from first-round matches of two professional tennis tournaments, the Australian Open and the French Open. Relevant prize dimensions are money and the ATP ranking points (career concerns). The analysis shows that both reward components shape players’ incentives to compete. If one neglects the points (non-monetary) dimension, the model underestimates heterogeneity and overestimates the incentive effect of monetary prizes. This result highlights the direction to improve reduced-form tests of the contest theory that always overlook the multi-dimensionality aspect.

Counterfactual experiments illustrate the existence of alternative prize allocations that can improve aggregate effort. On average, the French Open (the Australian Open) would benefit from increasing (decreasing) monetary rewards for first-round losers. If the managers of both contests were allowed to change the ranking points allocation, they would prefer zero losing prizes in this good. The difference stems from the fact that the French Open tends to couple more heterogeneous players. This pattern can be explained with contest-specific matching policies.

Redistributing 5% of money and 2% of ranking points from final winning rewards to first-round losing prizes could improve expected aggregate effort in matches with strong heterogeneity by more than 4.9% (3.4%) for the French Open (the Australian Open). Compared to the “Winner–Takes–All” allocation, the gain would be 5.6% (3.7%) in the French Open (the Australian Open). On the contrary, relatively even matches never benefit from positive losing prizes. These findings go in line with the theory and can be used in the job promotion setting. For example, when managers must distribute monetary and non-monetary prizes between heterogeneous workers, they may gain if assign positive losing benefits. If competitors have similar skills and / or preferences, it is optimal to give everything to the winner.

To conclude, the paper highlights the importance of multi-dimensional incentives in asymmetric contests, both in theoretical and empirical terms. The presence of additional reward items helps the designer to mitigate the negative effect of strong heterogeneity on effort exertion, if the prize for a loser increases. The structural analysis has confirmed that at least two goods (namely,\(^7\) Though, even with the convex cost specification it would never be possible to support the optimality of non-zero benefits for the last loser under single-item rewards.
money and career concerns) shape the incentives of workers (professional athletes) in job promotion (professional tennis competitions). These findings indicate that multi-dimensional preferences can affect the contestants’ behavior and, as a consequence, the optimal prize allocation significantly.

In the end, I indicate how this work could foster future research. First, one can introduce more than two players and prize items. This modified version would characterize a wider range of job promotion interactions and other contest-type settings. Another way to extend the framework is to look at multi-stage bilateral elimination contests with heterogeneous participants. In this case, the designer may want to maximize not only the total effort but also the winning probabilities of stronger players, and this results in a trade-off. To solve the problem, the designer can use two instruments: the prize allocation and the matching policy. The current empirical application highlights the importance of these two aspects to the contest setting. Finally, in terms of structural estimation, it would be interesting to analyze female players whose preferences may differ from those of males and discover if there are any gender-specific patterns.

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72 At the earlier stages of professional tennis tournaments managers care not only about aggregate effort. They also want stronger players to be advanced in the contest with a higher probability. This is the reason why seeding policies exist.
References


Tables and Figures

Figure 5: Empirical Distribution of Contestants’ Ranking Points

Table 6: The Australian Open Monetary Reward Schedules (A$1,000, CPI Adjusted), 2009–2015

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Round</td>
<td>29.2</td>
<td>23.2</td>
<td>24.1</td>
<td>17.4</td>
<td>17.8</td>
<td>18.4</td>
<td>18.5</td>
</tr>
<tr>
<td>2 Round</td>
<td>50.8</td>
<td>38.7</td>
<td>39.7</td>
<td>28.9</td>
<td>28.4</td>
<td>29.7</td>
<td>29.5</td>
</tr>
<tr>
<td>3 Round</td>
<td>82.5</td>
<td>58.1</td>
<td>61.9</td>
<td>47.6</td>
<td>48.4</td>
<td>49.1</td>
<td>48.6</td>
</tr>
<tr>
<td>4 Round</td>
<td>148</td>
<td>104.4</td>
<td>109.1</td>
<td>95.1</td>
<td>82.6</td>
<td>83.9</td>
<td>83.9</td>
</tr>
<tr>
<td>1/4 Final</td>
<td>287.6</td>
<td>209</td>
<td>218.1</td>
<td>190.2</td>
<td>186.6</td>
<td>188.7</td>
<td>173.7</td>
</tr>
<tr>
<td>1/2 Final</td>
<td>549.8</td>
<td>418.1</td>
<td>436.2</td>
<td>380.5</td>
<td>373.2</td>
<td>377.4</td>
<td>347.9</td>
</tr>
<tr>
<td>Runner-Up</td>
<td>1,311.1</td>
<td>1,025.4</td>
<td>1,059.9</td>
<td>1,001.3</td>
<td>977.4</td>
<td>990.7</td>
<td>953.1</td>
</tr>
<tr>
<td>Winner</td>
<td>2,622.2</td>
<td>2,049.9</td>
<td>2,119.9</td>
<td>2,002.5</td>
<td>1,954.9</td>
<td>1,981.4</td>
<td>1,906.1</td>
</tr>
<tr>
<td>Total Prize Money</td>
<td>33,834.6</td>
<td>28,388.1</td>
<td>26,172.2</td>
<td>21,766.7</td>
<td>22,214.2</td>
<td>22,733.7</td>
<td>22,053.6</td>
</tr>
<tr>
<td>Prize Spread</td>
<td>131.9</td>
<td>97.9</td>
<td>101.8</td>
<td>91.6</td>
<td>88</td>
<td>89.3</td>
<td>85.3</td>
</tr>
</tbody>
</table>

Note: prize spreads are calculated as the difference between the continuation value of winning round 1 and the losing prize at that stage.
Table 7: Growth Rates in Monetary Rewards for the Australian Open (%), 2009–2015

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Round</td>
<td>25.64</td>
<td>-3.54</td>
<td>38.28</td>
<td>-2.01</td>
<td>-3.41</td>
<td>-.49</td>
</tr>
<tr>
<td>2 Round</td>
<td>31.10</td>
<td>-2.48</td>
<td>36.91</td>
<td>1.97</td>
<td>-4.33</td>
<td>.60</td>
</tr>
<tr>
<td>3 Round</td>
<td>42.03</td>
<td>-6.25</td>
<td>30.24</td>
<td>-1.79</td>
<td>-1.30</td>
<td>.94</td>
</tr>
<tr>
<td>4 Round</td>
<td>41.74</td>
<td>-4.23</td>
<td>14.65</td>
<td>15.11</td>
<td>-1.59</td>
<td>.13</td>
</tr>
<tr>
<td>1/4 Final</td>
<td>37.58</td>
<td>-4.15</td>
<td>14.65</td>
<td>1.95</td>
<td>-1.12</td>
<td>8.64</td>
</tr>
<tr>
<td>1/2 Final</td>
<td>31.51</td>
<td>-4.15</td>
<td>14.65</td>
<td>1.95</td>
<td>-1.12</td>
<td>8.50</td>
</tr>
<tr>
<td>Runner-Up</td>
<td>27.86</td>
<td>-3.26</td>
<td>5.86</td>
<td>2.44</td>
<td>-1.34</td>
<td>3.95</td>
</tr>
<tr>
<td>Winner</td>
<td>27.91</td>
<td>-3.30</td>
<td>5.86</td>
<td>2.44</td>
<td>-1.34</td>
<td>3.95</td>
</tr>
<tr>
<td>Total Prize Money</td>
<td>19.19</td>
<td>8.47</td>
<td>20.24</td>
<td>-2.01</td>
<td>-2.29</td>
<td>3.08</td>
</tr>
<tr>
<td>Prize Spread</td>
<td>22.92</td>
<td>5.12</td>
<td>16.95</td>
<td>-0.52</td>
<td>-1.84</td>
<td>3.62</td>
</tr>
</tbody>
</table>

Note: prize spreads are calculated as the difference between the continuation value of winning round 1 and the losing prize at that stage.

Table 9: Estimated Heterogeneity: Contest-Specific Effect

<table>
<thead>
<tr>
<th>The Australian Open</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Money</td>
<td>.109</td>
<td>.087</td>
<td>1E–11</td>
<td>.53</td>
</tr>
<tr>
<td>Points</td>
<td>.23</td>
<td>.69</td>
<td>4.2E–05</td>
<td>.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>The French Open</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Money</td>
<td>.096</td>
<td>.077</td>
<td>1E–11</td>
<td>.58</td>
</tr>
<tr>
<td>Points</td>
<td>.2</td>
<td>.64</td>
<td>1.7E–05</td>
<td>.77</td>
</tr>
</tbody>
</table>

Note: for every match $j$ heterogeneity is measured as the absolute difference in contestants’ skills and valuations:

$$\text{het}_j^A = |\hat{\alpha}(x_{ij}) - \hat{\alpha}(x_{-ij})|, \quad \text{het}_j^B = |\hat{\beta}(x_{ij}) - \hat{\beta}(x_{-ij})|$$

where $\hat{\alpha}(\cdot) = c(\cdot) \alpha(\cdot)$ and $\hat{\beta}(\cdot) = c(\cdot) \beta(\cdot)$. 
Table 8: Summary Statistics: Both Contests in 2009–2015

<table>
<thead>
<tr>
<th></th>
<th>$N$</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monetary Losing Prize (A$1,000)</td>
<td>1680</td>
<td>25.94</td>
<td>61.79</td>
<td>19.4</td>
<td>38.25</td>
</tr>
<tr>
<td>Monetary Winning Prize (A$1,000)</td>
<td>1680</td>
<td>2,171.5</td>
<td>396.55</td>
<td>1,605.38</td>
<td>3,100</td>
</tr>
<tr>
<td>Prize Spread (A$1,000)</td>
<td>1680</td>
<td>110.79</td>
<td>23.22</td>
<td>85.31</td>
<td>155.97</td>
</tr>
<tr>
<td>Total Prize Money (A$1,000)</td>
<td>1680</td>
<td>2.65E+04</td>
<td>4,752.56</td>
<td>2.18E+04</td>
<td>3.57E+04</td>
</tr>
<tr>
<td>Number of Unforced Errors per Set Played</td>
<td>1680</td>
<td>8.26</td>
<td>5.54</td>
<td>0</td>
<td>31</td>
</tr>
<tr>
<td>Age</td>
<td>1680</td>
<td>27.22</td>
<td>3.66</td>
<td>17</td>
<td>38</td>
</tr>
<tr>
<td>Body Mass Index (BMI)</td>
<td>1680</td>
<td>23.13</td>
<td>1.29</td>
<td>19.24</td>
<td>26.85</td>
</tr>
<tr>
<td>Ranking Points</td>
<td>1680</td>
<td>1249.29</td>
<td>1694.6</td>
<td>10</td>
<td>14,960</td>
</tr>
<tr>
<td>$\text{het}_j^P$</td>
<td>840</td>
<td>1342.54</td>
<td>2150.17</td>
<td>0</td>
<td>14,162</td>
</tr>
<tr>
<td>Betting Odd</td>
<td>1680</td>
<td>3.67</td>
<td>5.14</td>
<td>1</td>
<td>61</td>
</tr>
<tr>
<td>$\text{het}_j^B$</td>
<td>840</td>
<td>4.63</td>
<td>6.62</td>
<td>0</td>
<td>59.99</td>
</tr>
</tbody>
</table>

Dummy Variables

<table>
<thead>
<tr>
<th></th>
<th>$N$</th>
<th>Frequency of 1’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Seed}_i = 1$ if player $i$ is seeded</td>
<td>1680</td>
<td>.25</td>
</tr>
<tr>
<td>$\text{Hbias}_i = 1$ if player $i$ has a home bias</td>
<td>1680</td>
<td>.18</td>
</tr>
<tr>
<td>$\text{Tour} = 1$ for the Australian Open</td>
<td>1680</td>
<td>.5</td>
</tr>
</tbody>
</table>

Note: $\text{het}_j^P = |\text{points}_{ij} - \text{points}_{-ij}|$ ($\text{het}_j^B = |\text{Bet}_{ij} - \text{Bet}_{-ij}|$) approximates heterogeneity in match $j$ as the absolute difference in contestants’ ranking points (betting odds).
Figure 6: Goodness-of-Fit: Age- and Points-Based Groups & One-Dimensional Prizes

Table 10: Estimated Heterogeneity: No Contest Specific Effect

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The Australian Open</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Money</td>
<td>.091</td>
<td>.073</td>
<td>1.4E–11</td>
<td>.44</td>
</tr>
<tr>
<td>Points</td>
<td>.17</td>
<td>.53</td>
<td>1.7E–05</td>
<td>.58</td>
</tr>
<tr>
<td><strong>The French Open</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Money</td>
<td>.096</td>
<td>.077</td>
<td>1E–11</td>
<td>.58</td>
</tr>
<tr>
<td>Points</td>
<td>.2</td>
<td>.64</td>
<td>1.7E–05</td>
<td>.77</td>
</tr>
</tbody>
</table>

Note: for every match \( j \) heterogeneity is measured as the absolute difference in contestants’ skills and valuations:

\[
het^A = |\tilde{\alpha}(x_{1j}) - \tilde{\alpha}(x_{2j})|, \quad het_B = |\tilde{\beta}(x_{1j}) - \tilde{\beta}(x_{2j})|
\]

where \( \tilde{\alpha}(\cdot) = c(\cdot)\alpha(\cdot) \) and \( \tilde{\beta}(\cdot) = c(\cdot)\beta(\cdot) \).
Figure 7: Mean Expected Effort: the Australian Open and Fixed Prizes in Dimension $B$

Note: $K = 821$; the total number of simulations is 1,000; solid lines with bubbles correspond to a 0.5–quantile of mean expected effort simulated distributions; lower and upper confidence bounds reflect 0.16– and 0.84–quantiles of mean expected effort simulated distributions; the grid for monetary losing prizes is $\bar{A}_L = [0, \bar{u} A_u^L]$ with $\bar{u} = 6$ and 21 steps.
Figure 8: Mean Expected Effort: the French Open and Fixed Prizes in Dimension $B$

Note: $K = 821$; the total number of simulations is 1,000; solid lines with bubbles correspond to a 0.5–quantile of mean expected effort simulated distributions; lower and upper confidence bounds reflect 0.16– and 0.84–quantiles of mean expected effort simulated distributions; the grid for monetary losing prizes is $\bar{A}^L = [0, \bar{u}A^L]$ with $\bar{u} = 5$ and 21 steps.
Figure 9: Group-Specific Improvement in Mean Expected Effort with respect to $A^L = 0$ (%): the Australian Open and Fixed Prizes in Dimension $B$

**Table 11: Expected Aggregate Effort with One- and Two-Dimensional Rewards: Actual Prize Allocation**

<table>
<thead>
<tr>
<th></th>
<th>The Australian Open</th>
<th>The French Open</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Prizes in Ranking Points</td>
<td>24.96</td>
<td>23.49</td>
</tr>
<tr>
<td>Two-Dimensional Prizes</td>
<td>48.02</td>
<td>42.12</td>
</tr>
</tbody>
</table>

Note: total number of simulations is 1,000; lower and upper confidence bounds reflect 0.16- and 0.84-quantiles of mean expected aggregate effort simulated distribution.
Figure 10: Group-Specific Improvement in Mean Expected Effort with respect to $A^L = 0$ (%): the French Open and Fixed Prizes in Dimension $B$

Note: $H^p_j = |\text{points}_{ij} - \text{points}_{-ij}|$ for every match $j$; $K = 821$; the total number of simulations is 1,000; solid lines with bubbles correspond to a 0.5-quantile of mean expected effort simulated distributions; the grid for monetary losing prizes is $A^L = [0, \bar{u} A^L]$ with $\bar{u} = 5$ and 21 steps.
Figure 11: Contest-Specific Mean Expected Aggregate Effort: Flexible Reward Schedules

Flexible $A^L$ and Optimal $B^L$: the Australian Open

Flexible $B^L$ and Optimal $A^L$: the Australian Open

Flexible $A^L$ and Optimal $B^L$: the French Open

Flexible $B^L$ and Optimal $A^L$: the French Open

Note: $K = 821$; the total number of simulations is 1,000; solid lines with bubbles correspond to a 0.5–quantile of mean expected effort simulated distributions; lower and upper confidence bounds reflect 0.16– and 0.84–quantiles of mean expected effort simulated distributions; the grids for monetary and non-monetary losing prizes are $\bar{A}^L = [0, \bar{u}A^L]$ and $\bar{B}^L = [0, \bar{u}B^L]$ with $\bar{u} = 5$ and 21 steps.
Table 12: Best Group-Specific Prize Schedules: The Australian Open

<table>
<thead>
<tr>
<th>Group</th>
<th>((A^L, B^L)) in The Best Prize Scheme</th>
<th>Highest Effort</th>
<th>Improvement in Effort w.r.t. ((0, 0))</th>
<th>Improvement in Effort w.r.t. ((A^L_a, B^L_a))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(H^P_j \in [0, 200])</td>
<td>((0, 0))</td>
<td>44.2</td>
<td>0%</td>
<td>1.2%</td>
</tr>
<tr>
<td>(H^P_j \in (200, 500])</td>
<td>((0, 0))</td>
<td>44.8</td>
<td>0%</td>
<td>.8%</td>
</tr>
<tr>
<td>(H^P_j \in (500, 1000])</td>
<td>((116.4, 3))</td>
<td>39.6</td>
<td>1.1%</td>
<td>.9%</td>
</tr>
<tr>
<td>(H^P_j \in (1000, 1600])</td>
<td>((116.4, 42))</td>
<td>37.3</td>
<td>3.7%</td>
<td>3.4%</td>
</tr>
<tr>
<td>(H^P_j \in (1600, 3000])</td>
<td>((116.4, 60))</td>
<td>36.9</td>
<td>1.9%</td>
<td>1.8%</td>
</tr>
<tr>
<td>(H^P_j &gt; 3000)</td>
<td>((0, 0))</td>
<td>34.8</td>
<td>0%</td>
<td>.5%</td>
</tr>
</tbody>
</table>

Note: \(H^P_j = |points_{ij} - points_{-ij}|\) for every match \(j\); \(A^L\) is measured in A$1,000; \(B^L\) is measured in points; total number of simulations is 1,000; lower and upper confidence bounds reflect 0.16– and 0.84–quantiles of mean expected effort simulated distributions; the grids for monetary and non-monetary losing prizes are \(\bar{A}^L = [0, \bar{u}A^L_a]\) and \(\bar{B}^L = [0, \bar{u}B^L_a]\) with \(\bar{u} = 6\) and 21 steps.

Table 13: Best Group-Specific Prize Schedules: The French Open

<table>
<thead>
<tr>
<th>Group</th>
<th>((A^L, B^L)) in The Best Prize Scheme</th>
<th>Highest Effort</th>
<th>Improvement in Effort w.r.t. ((0, 0))</th>
<th>Improvement in Effort w.r.t. ((A^L_a, B^L_a))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(H^P_j \in [0, 200])</td>
<td>((0, 0))</td>
<td>45.1</td>
<td>0%</td>
<td>.5%</td>
</tr>
<tr>
<td>(H^P_j \in (200, 500])</td>
<td>((0, 0))</td>
<td>45.2</td>
<td>0%</td>
<td>.6%</td>
</tr>
<tr>
<td>(H^P_j \in (500, 1000])</td>
<td>((100.5, 32.5))</td>
<td>41.2</td>
<td>1.8%</td>
<td>1.6%</td>
</tr>
<tr>
<td>(H^P_j \in (1000, 1600])</td>
<td>((100.5, 32.5))</td>
<td>39.1</td>
<td>5.3%</td>
<td>4.8%</td>
</tr>
<tr>
<td>(H^P_j \in (1600, 3000])</td>
<td>((100.5, 50))</td>
<td>37.4</td>
<td>5.6%</td>
<td>4.9%</td>
</tr>
<tr>
<td>(H^P_j &gt; 3000)</td>
<td>((100.5, 47.5))</td>
<td>35.1</td>
<td>.4%</td>
<td>.5%</td>
</tr>
</tbody>
</table>

Note: \(H^P_j = |points_{ij} - points_{-ij}|\) for every match \(j\); \(A^L\) is measured in A$1,000; \(B^L\) is measured in points; total number of simulations is 1,000; lower and upper confidence bounds reflect 0.16– and 0.84–quantiles of mean expected effort simulated distributions; the grids for monetary and non-monetary losing prizes are \(\bar{A}^L = [0, \bar{u}A^L_a]\) and \(\bar{B}^L = [0, \bar{u}B^L_a]\) with \(\bar{u} = 5\) and 21 steps.
Appendix A. Theoretical Model: Extensions

Bundling vs. Two Simultaneous Contests

Suppose there exists an alternative way to use both goods. The designer can run two separate contests with one-dimensional rewards instead of a single competition with bundled prizes. If expected aggregate effort taken over former contests always exceeds the one induced in the latter case, results of Theorem 1 break down easily.

Assume the designer has two possibilities:

1. Scheme $b$: run one contest, allocate prize bundles and induce expected aggregate effort $J^b(A^W, A^L, B^W, B^L)$.

2. Scheme $ub$: run two separate competitions with single-item prizes and take a sum of expected aggregate effort over them:

$$J^{ub}(A^W, A^L, B^W, B^L) = J(A^W, A^L, 0, 0) + J(0, 0, B^W, B^L)$$

where function $J(\cdot)$ was defined in the original model.

Let $W^b = \left( \begin{array}{c} A^W_b \\ B^W_b \end{array} \right)$ and $L^b = \left( \begin{array}{c} A^L_b \\ B^L_b \end{array} \right)$, $W^{ub} = \left( \begin{array}{c} A^W_{ub} \\ B^W_{ub} \end{array} \right)$ and $L^{ub} = \left( \begin{array}{c} A^L_{ub} \\ B^L_{ub} \end{array} \right)$ be optimal prizes for scheme $b$ ($ub$). In a single-item contest, the designer always assigns the highest possible prize spread (see the proof of Proposition 2, Lemma 5):

$$W^{ub} = \left( \begin{array}{c} 1 \\ 1 \end{array} \right), \quad L^{ub} = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$$

Theorem 1 characterizes optimal reward schedules for scheme $b$. Abusing notations, I introduce $J^l(W^l, L^l) = J^l(A^W_l, A^L_l, B^W_l, B^L_l)$, $l = \{b, ub\}$. When optimal prize schedules are known, the designer faces the following problem:

$$\max \{ J^b(W^b, L^b), J^{ub}(W^{ub}, L^{ub}) \}$$

As before, $R$ defines a set of all feasible valuation profiles:

$$R = \{ r : \alpha_i \geq 0, \beta_i \geq 0, i = \{g, n\} \}$$

Theorem 2 states the existence of valuation profiles such that scheme $b$ (with or without positive losing prizes) induces higher expected aggregate effort:

73Labels $b$ and $ub$ denote bundled and unbundled prize schedules, respectively.
Theorem 2. There exists a non-empty subset of $R, R^b$, such that for any $\{\alpha_g, \alpha_n, \beta_g, \beta_n\} \in R^b$ the designer prefers one contest with bundled prizes to two separate contests with unbundled prizes.

Proof. See Appendix E. \hfill \Box

Set $R^b$ contains two types of valuation profiles. The first type supports optimal bundles where a winner takes all. These profiles must be such that the greediest player values only one good more than the opponent ($\alpha_g > \alpha_n, \beta_g < \beta_n$ or vise versa). The result has the following interpretation. Suppose player $g$ likes both items more ($\alpha_g > \alpha_n$ and $\beta_g > \beta_n$). The “Winner–Takes–All” bundle is optimal when contestants have relatively homogeneous preferences over both dimensions. Since player $g$ values goods more, his relative power is higher in the competition with bundled prizes than in two separate contests. As a result, the opponent has less incentives to exert effort in the former case. Formally, for $\alpha_g > \alpha_n, \beta_g > \beta_n$ player $n$ chooses $e > 0$ with a lower probability when the designer runs a single contest and allocates the “Winner–Takes–All” bundle:

$$P(e_n > 0 | b, WTA) < P(e_n > 0 | ub) \text{ if } \alpha_g > \alpha_n, \beta_g > \beta_n$$

If contestant $g$ likes only one good more, the inequality can change its sign. Then, player $n$ exerts strictly positive effort with a higher probability when the designer allocates the “Winner–Takes–All” bundle. Given that participants have relatively homogeneous preferences, two separate competitions with one-dimensional prizes induce sufficient expected aggregate effort. Thus, the designer prefers scheme $ub$ over $b$ when player $g$ values both goods more. Otherwise, he can run one contest and allocate the “Winner–Takes–All” bundle.

Another type of valuation profiles entering $R^b$ supports optimal bundles with positive losing prizes. A sufficient condition to make this reward schedule optimal is strong heterogeneity in players’ preferences. This also means that single-item contests are very uneven and result in low expected aggregate effort. Two-dimensional prizes with positive losing benefits help the designer to mitigate a negative effect players’ heterogeneity has on effort exertion. To illustrate this point, I provide a numerical example.

---

74I use Proposition 1 to calculate these probabilities:

$$P(e_n > 0 | b, WTA) = \frac{\alpha_n + \beta_n}{\alpha_g + \beta_g}$$

$$P(e_n > 0 | ub) = \frac{\alpha_n \beta_n}{\alpha_g \beta_g} + \frac{\alpha_n}{\alpha_g} \left(1 - \frac{\beta_n}{\beta_g}\right) + \frac{\beta_n}{\beta_g} \left(1 - \frac{\alpha_n}{\alpha_g}\right)$$

where the latter is a probability that player $n$ exerts positive effort at least in one contest with single-item prizes. Then:

$$P(e_n > 0 | b, WTA) < P(e_n > 0 | ub) \iff -\alpha_n \beta_g (\beta_g - \beta_n) - \beta_n \alpha_g (\alpha_g - \alpha_n) < 0$$

and the inequality always holds for $\alpha_g > \alpha_n, \beta_g > \beta_n$.

75For more details on this mechanism, see Subsection 3.3.1.
**Example.** Assume $\left( \frac{\alpha_1}{\beta_1} \right) = \begin{pmatrix} 10 \\ 8 \end{pmatrix}$ and $\left( \frac{\alpha_2}{\beta_2} \right) = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$. I start from scheme $b$. The “Winner–Takes–All” schedule results in $J^b(1, 0, 1, 0) = 7.8$. Player 1 is the “greediest-to-win” ($g = 1$). For given $\alpha_g$ and $\alpha_n$, it is always optimal to assign the highest prize spread in good $A$ (see the proof of Theorem 1). Type $g$ is more sensitive to incentives in dimension $B$ ($\frac{\alpha_g}{\beta_g} < \frac{\alpha_n}{\beta_n}$). Then, positive losing benefits can appear only in dimension $B$. In fact, the optimal allocation of item $B$ for specified preferences is $B^W = \frac{3}{7}, B^L = \frac{4}{7}$ (see the proof of Theorem 1). This alternative prize schedule induces $J^b(1, 0, \frac{3}{7}, \frac{4}{7}) = 8.9$ and dominates the “Winner–Takes–All” scheme. If the designer runs two separate single-item contests, he gets $J^{ub}(1, 0, 1, 0) = 9.1$. Thus, scheme $ub$ must be chosen. This is the case because heterogeneity in dimension $B$ is not strong enough, and the corresponding contest with single-item prizes generates enough effort to make scheme $ub$ beneficial.

Next, take the same $\left( \frac{\alpha_1}{\beta_1} \right)$ combined with $\left( \frac{\alpha_2}{\beta_2} \right) = \begin{pmatrix} 9 \\ 0.1 \end{pmatrix}$. Now the difference between $\beta_1$ and $\beta_2$ increases. Again, player 1 is of type $g$, and his incentives show relatively more sensitivity in dimension $B$ ($\frac{\alpha_g}{\beta_g} < \frac{\alpha_n}{\beta_n}$). Consider scheme $b$. The “Winner–Takes–All” allocation induces $J^b(1, 0, 1, 0) = 6.9$. The optimal prize schedule for given preferences requires $B^W \approx \frac{11}{25}, B^L \approx \frac{14}{25}$, and expected aggregate effort is $J^b(1, 0, \frac{11}{25}, \frac{14}{25}) \approx 9$. When the designer implements scheme $ub$, he gets $J^{ub}(1, 0, 1, 0) = 8.6$. As a result, when heterogeneity in dimension $B$ is very strong, the designer does not benefit from two contests with single-item rewards and prefers to create a bundle with a positive losing prize.

Overall, Theorem 2 strengthens the results of the original model: bundling with positive losing prizes can be beneficial even if the designer has a freedom to run separate contests over two dimensions.

**Asymmetric Information about Contestants’ Types**

In this section, I introduce asymmetric information about contestants’ types. This extension is important for two reasons. First, players often have unobservable characteristics that can affect their effort choices. Second, in many real-life contests the designer must commit to the prize allocation before he learns exact matching. For example, contest organizers in professional tennis announce monetary reward schedules before the final draw is known.

Suppose all key assumptions of the original model hold. However, now $\alpha_i$ is private information of player $i$:

$$\alpha_i = \{ \alpha_i, \bar{\alpha}_i \}, \quad 0 < \alpha_i < \bar{\alpha}_i, \quad i = \{1, 2\}$$

where realizations of $\alpha_1$ and $\alpha_2$ are independent, $P(\alpha_i = \bar{\alpha}_i) = k \forall i = \{1, 2\}$. For tractability
reasons, $\beta_1$ and $\beta_2$ are assumed to be common knowledge. This allows me to restrict a number of types every contestant has by two and avoid excessive parametrization.\footnote{Equivalently, one could assume private information about $\beta_i$ but state that two random variables are perfectly correlated:}

Without loss of generality, I fix $\beta_1 > \beta_2$.

Define $\bar{U}_k^i = \alpha_i A^k + \beta_i B^k$ and $U_k^i = \alpha_i A^k + \beta_i B^k$ if player $i$ wins prize $k$. As every contestant has two types, I introduce the following definition:

**Definition 4.** Let $\bar{\bar{t}}_i (\Delta_A, \Delta_B) = \bar{U}_W^i - \bar{U}_L^i \equiv \bar{\bar{t}}_i$ and $\bar{t}_i (\Delta_A, \Delta_B) = U_W^i - U_L^i \equiv \bar{t}_i$ be winning benefits of contestant $i$ with realized valuations $\bar{\alpha}_i$ and $\bar{\alpha}_i$, respectively. Define $t^g_i = \max \{\bar{\bar{t}}_i, \bar{t}_i\}$ and $t^n_i = \min \{\bar{\bar{t}}_i, \bar{t}_i\}$. Then $\bar{\bar{t}}_i$ is the "greediest-to-win" type of contestant $i$ ($g = i$) if and only if $\bar{\bar{t}}_i = t^g_i$; otherwise, $\bar{\bar{t}}_i$ is not the "greediest-to-win" type of contestant $i$ ($n = i$).

I do not compare types between contestants but identify two states the same player can face. As before, every participant chooses his effort taking into account announced prizes and the competitor’s action. When I analyze the game between contestants, I look at a particular class of equilibria.

**Definition 5.** Let $e^* (t^j_i)$ be effort type $j$ of contestant $i$ exerts in equilibrium. The equilibrium is in monotonically increasing strategies if and only if $e^* (t^g_i) \geq e^* (t^n_i)$ for any $i = \{1, 2\}$.

The definition says that in equilibrium $g$-types associated with higher winning benefits must never exert less effort than $n$-types. Also, I require non-triviality of equilibria:

**Definition 6.** The equilibrium is trivial if and only if at least one $g$-type chooses $e = 0$ with probability 1. Otherwise, the equilibrium is non-trivial.

Let $G_i (t^j_i, e)$ be a probability that type $j$ of contestant $i$ chooses at most $e$, and $s^j_i$ denotes a support of $G_i (t^j_i, e)$. Proposition 4 characterizes contestants’ equilibrium behavior:

**Proposition 4.** For $\min \{t^n_1, t^n_2\} > 0$ there exists a unique non-trivial equilibrium in monotonically increasing strategies such that:

- At least one type places an atom at zero: $\exists i = \{1, 2\}, j = \{g, n\} : G_i (t^j_i, 0) > 0$;

- There is no $e > 0$ played with a positive probability;

\footnote{Equivalently, one could assume private information about $\beta_i$ but state that two random variables are perfectly correlated:}

\[\beta_i = \left\{\bar{\beta}_i, \bar{\beta}_i\right\}, 0 < \bar{\beta}_i < \bar{\beta}_i\]

\[P (\beta_i = \bar{\beta}_i | \bar{\alpha}_i) = 1, P (\beta_i = \bar{\beta}_i | \bar{\alpha}_i) = 1\]

\[P (\beta_i = \bar{\beta}_i) = P (\alpha_i = \bar{\alpha}_i) = k \forall i = \{1, 2\}\]
• Supports of $G_i(t^g_i, e)$, $i = \{1, 2\}$ have the same supremum: $\sup (s^g_1) = \sup (s^g_2)$;

• Supports of $G_i(t^n_i, e)$, $i = \{1, 2\}$ have the same infimum, and it is equal to zero: $\inf (s^n_1) = \inf (s^n_2) = 0$.

Proof. See Appendix E.

To construct this equilibrium, I use an algorithm developed by Siegel (2014) for asymmetric all-pay auctions with two participants, private information and discrete types. I extend this characterization to the case of perfectly divisible multiple goods and introduce prize-dependent types.

When the second strongest type has a non-positive winning benefit ($\min \{t^g_1, t^g_2\} \leq 0$), the equilibrium becomes trivial. Otherwise, $g$-types of both contestants choose $e > 0$ with a strictly positive probability. As before, the equilibrium features mixed strategies. All effort choices must belong to $s = [0, \min \{t^g_1, t^g_2\}]$. The equilibrium partitions $s$ into intervals of different length. On every interval, particular types of contestants 1 and 2 compete by randomizing uniformly. The equilibrium partition depends on the prize structure ($W$ and $L$), the probability distribution ($k$) and includes up to 3 intervals. One can find the greediest competitor for every element of the equilibrium partition:

Definition 7. Define order $Q$ over elements of the equilibrium partition where $q = 1$ corresponds to $e = 0$. Let $t_{i,q}$ be a type of contestant $i$ playing on interval $q \in Q$. Then, $t_{i,q}$ is the “greediest-to-win” on interval $q \in Q$ ($g_q = i$) if and only if $t_{i,q} = \max \{t_{1,q}, t_{2,q}\}$.

Using these notations, I define a relative power of type $g_q$ in this setting:

Definition 8. A relative power of contestant $g_q$ on interval $q \in Q$, $q > 1$ is a probability that his competitor plays on interval $q - 1 \in Q$.

This notion of the relative power is very similar to one introduced for the case of symmetric information. In the lowest interval of the equilibrium partition ($q = 2$), the relative power of player $g_q$ is just a probability that the opponent chooses $e = 0$. As in the original model, two-dimensional prizes allow the designer to affect the equilibrium partition by changing winning and losing prizes. The mechanism driving this result was described in Subsection 3.2.

The designer chooses the prize allocation that maximizes expected aggregate effort, $J_k(\cdot)$, given feasibility constraints. Let $r_k$ be a probability–valuation profile, and $R_k$ denotes a set of all feasible $r_k$’s:

$$R_k = \{r_k : \alpha_i \geq 0, \bar{\alpha}_i \geq 0, \beta_i \geq 0, k \in [0, 1]\}$$

The argument is exactly the same as the one provided for the case of $t_n \leq 0$ in Proposition 1.

In case of single-item rewards, this value is constant.
To simplify the analysis, I fix two patterns in players’ preferences. First, I assume that contestant 1 has stronger willingness to win for both realizations of $\alpha_1$ and $\alpha_2$ when the “Winner–Takes–All” schedule is implemented:

$$t^1_j (1, 1) > t^2_j (1, 1) \forall j = \{g, n\} \iff \alpha_1 > \bar{\alpha}_2 + \beta_1 - \beta_2$$

Second, I look at symmetric profiles:

**Definition 9.** Probability–valuation profile $r_k$ is symmetric if and only if $\min \{\alpha_1, \alpha_2\} > \beta_1$ or $\max \{\bar{\alpha}_1, \bar{\alpha}_2\} < \beta_2$, $\beta_1 > \beta_2$. Otherwise, probability–valuation profile is asymmetric. $R^*_k \subset R_k$ is a set of all symmetric probability–valuation profiles.

With symmetric profiles, both contestants enjoy the same good more for all realizations of $\alpha_1$ and $\alpha_2$. Proposition 5 states the existence of probability–valuation structures that satisfy all imposed constraints and make positive losing prizes optimal.\(^{79}\)

**Proposition 5.** For $\beta_1 > \beta_2$ and $\alpha_1 > \bar{\alpha}_2 + \beta_1 - \beta_2$ there exists a non-empty subset of $R^*_k$, $R^L_k$, such that for any probability–valuation profile in $R^L_k$, the designer uses both goods completely and assigns a positive losing prize in dimension A.

**Proof.** See Appendix E. \(\square\)

Mechanisms driving the optimality of positive losing prizes are exactly the same as described in Subsection 3.3.1. Higher losing rewards lower winning benefits and, consequently, incentives to exert effort (direct effect). At the same time, it reduces the advantage of the greediest player on every interval of the equilibrium partition and, consequently, incentivizes the opponent to compete more (equilibrium effect). When contestants’ preferences are very heterogeneous, the latter effect dominates, and expected aggregate effort goes up.

Overall, key results of the baseline model hold even when asymmetric information is introduced, and it makes the proposed framework robust in this respect.

\(^{79}\)Since the result concerning the optimality of positive losing prizes is the most striking one, I concentrate on proving only this case and show that similar mechanisms work under asymmetric information. I do not analyze the optimality of the “Winner–Takes–All” bundle separately although there exist probability–valuation profiles supporting it.
Appendix B. The Effort Proxy Performance in the Reduced-Form Tests

This section shows that the proposed measure based on a number of unforced errors per set played can approximate contestants' effort well. I use the unbalanced panel of 320 players participated either in the Australian Open or the French Open in 2009–2015. Following the discussion in Section 4, I do not address selection issues here.

Since skills and preferences cannot be recovered in the reduced-form setting, one must approximate players' heterogeneity. Following methodology developed in other empirical tests of the contest theory, I measure heterogeneity by taking the absolute difference in ranking points of the competitors (variable $het^P_j$ for game $j$).\footnote{For instance, see Sunde (2009).}

Let $A$ and $B$ denote money and ranking points (career concerns), respectively. I characterize first-round winning prizes as a continuation value of being advanced in the contest. Suppose success and failure at later stages of the tournament are equally probable. Also, assume identities of potential future opponents do not matter.\footnote{This strategy was used in several empirical papers (for instance, see Silverman and Seidel (2011), Ivankovic (2007)) with the following argument: winning and losing probabilities of different players average to “1/2–1/2”.

With this approach, prize spreads are the same for all players. As a result, heterogeneity in first-round games stems only from different skills and asymmetric preferences.

Following previous reduced-form tests of the contest theory, in this section I consider only monetary incentives. Standard tournament models predict that good effort proxies must reflect the following patterns:

1. Effort increases in the prize spread ($\Delta_A = A^W - A^L$) (Hypothesis 1, or $H_1$);

2. Effort decreases in contestants' heterogeneity ($het^P_j$) (Hypothesis 2, or $H_2$).

In addition, I want to see how monetary losing prizes affect effort choices. However, with the player-invariant spread specification this is not possible because the two values are strongly correlated ($corr (A^L, \Delta_A) = 0.92$). Table 15 contains estimation results. $H_1$ and $H_2$ cannot be rejected in all specifications: individual effort increases (decreases) when monetary prize spreads (heterogeneity) grow. In this respect, the proposed measure behaves as a good proxy for players' effort.

To capture the difference in contestants' preferences, I introduce another specification of the prize spread. Players are divided into six groups based on their ranking. For each of them, I calculate empirical frequencies of being advanced to a particular contest level (see Table 14). The algorithm generates non-parametric group-specific continuation values and prize spreads ($\Delta^G_A$). Now, monetary losing benefits can also be used as an explanatory variable ($corr (A^L, \Delta^G_A) = 0.07$).
When I analyze all matches together, $H_1$ cannot be rejected (see Table 16, specifications (1)–(3)), and the candidate effort measure performs well. However, there are some interesting features to be emphasized. First, monetary losing prizes matter for contestants’ effort exertion, and the effect is positive. Second, there exists the group of participants whose effort decreases in prize spreads but grows in losing benefits (see specification (4) in Table 16). $H_2$ finds support in all specifications.

Finally, one can argue more unforced errors stem from risk-taking behavior but not lower effort.\[^{82}\] To address this point, I suggest the following approach. I assume that contestants taking more risk hit the ball stronger. As a result, their serving speed increases. Then if the number of unforced errors per set also includes risk-taking, it must be positively correlated with the latter variable. Table 17 shows that relevant regression coefficients are insignificant. As a result, the proposed proxy does not reveal evidence of risk-taking.

Table 14: Round–Specific Empirical Frequencies of Winning for Different Contestants’ Groups

<table>
<thead>
<tr>
<th></th>
<th>$l = 1$</th>
<th>$l = 2$</th>
<th>$l = 3$</th>
<th>$l = 4$</th>
<th>$l = 5$</th>
<th>$l = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Round 2</td>
<td>1</td>
<td>1</td>
<td>0.95</td>
<td>0.62</td>
<td>0.32</td>
<td>0.14</td>
</tr>
<tr>
<td>Round 3</td>
<td>1</td>
<td>0.93</td>
<td>0.92</td>
<td>0.41</td>
<td>0.17</td>
<td>0.38</td>
</tr>
<tr>
<td>Round 4</td>
<td>1</td>
<td>0.85</td>
<td>0.68</td>
<td>0.22</td>
<td>0.2</td>
<td>0</td>
</tr>
<tr>
<td>1/4 Final</td>
<td>0.73</td>
<td>0.73</td>
<td>0.39</td>
<td>0.33</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1/2 Final</td>
<td>0.88</td>
<td>0.5</td>
<td>0.33</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Final</td>
<td>0.71</td>
<td>0.25</td>
<td>0.33</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Note:

1. Cells of the table contain empirical frequencies of group $l$ (column) winning round $j$ (row) in the Australian Open or the French Open in 2009–2015, $i = \{1, ..., 6\}$, $j = \{2, 3, 4, 1/4 \text{final}, 1/2 \text{final}, \text{final}\}$.

2. Groups based on ranking points look as follows:

\[
i \in l \iff \text{points}_i \in [G_{7-l}, G_{7-l+1}], \ l = \{1, ..., 6\}
\]

\[
G_k \in G, \ G = \{0, 500, 1000, 3000, 6000, 10000, \max(\text{points}_i)\}
\]

3. One can calculate group-specific prize spreads using empirical frequencies of winning and reward schedules.

[^82]: Since the theoretical setup assumes risk-neutrality, I prefer to isolate this channel.
Table 15: Individual Effort: Fixed–Effect Specification and Constant Prize Spreads

<table>
<thead>
<tr>
<th></th>
<th>(1) No Other Controls</th>
<th>(2) Game and Tournament Controls–I</th>
<th>(3) Game and Tournament Controls–II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_{A,t}$</td>
<td>0.048***</td>
<td>0.052***</td>
<td>0.05***</td>
</tr>
<tr>
<td>A$$1000</td>
<td>(0.017)</td>
<td>(0.017)</td>
<td>(0.017)</td>
</tr>
<tr>
<td>$het_{ij}$</td>
<td>-0.4E–03***</td>
<td>-0.5E–03***</td>
<td>-0.5E–03***</td>
</tr>
<tr>
<td></td>
<td>(0.7E–0.4)</td>
<td>(0.8E–0.4)</td>
<td>(0.8E–0.4)</td>
</tr>
<tr>
<td>$A^t_{j}$</td>
<td>No</td>
<td>No</td>
<td>-5.21</td>
</tr>
<tr>
<td>Match-Spec.</td>
<td>No</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>Contest-Spec.</td>
<td>No</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>Time Effects</td>
<td>Yes</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>Trend</td>
<td>Yes</td>
<td>Yes</td>
<td></td>
</tr>
</tbody>
</table>

$$N = 1647$$  $$F(9, 532) = 8.42$$  $$N = 1642$$  $$F(10, 532) = 7.33$$  $$N = 1642$$  $$F(15, 531) = 6.84$$

Note: dependent variable – individual effort measured as $\tilde{e}_{ij} = \hat{u} - \tilde{u}_{ij}$ where $\tilde{u}_{ij}$ is a number of unforced errors per set player $i$ made in match $j$, $\hat{u} = \max_i \tilde{u}_{ij}$; standard errors are clustered over players; *, **, and *** correspond to 90%, 95%, and 99% significance; other controls include seeding policies of the contests, players’ and their competitors’ individual characteristics, betting odds, growth rates of $A_{t}$ and $\Delta_{A,t}$, total prize money.

Table 16: Individual Effort: Fixed–Effect Specification and Flexible Prize Spreads

<table>
<thead>
<tr>
<th></th>
<th>(1) No Other Controls</th>
<th>(2) Game and Tournament Controls–I</th>
<th>(3) Game and Tournament Controls–II</th>
<th>(4) Restricted Set of Players</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_{A,t}$</td>
<td>0.004***</td>
<td>0.0035**</td>
<td>0.0036**</td>
<td>-0.24**</td>
</tr>
<tr>
<td>A$$1000</td>
<td>(0.001)</td>
<td>(0.001)</td>
<td>(0.001)</td>
<td>(0.001)</td>
</tr>
<tr>
<td>$het_{ij}$</td>
<td>-0.5E–03***</td>
<td>-0.5E–03***</td>
<td>-0.5E–03***</td>
<td>-0.45E–03***</td>
</tr>
<tr>
<td></td>
<td>(0.8E–0.4)</td>
<td>(0.8E–0.4)</td>
<td>(0.9E–0.4)</td>
<td>(0.13E–0.4)</td>
</tr>
<tr>
<td>$A^t_{j}$</td>
<td>No</td>
<td>0.18**</td>
<td>0.19**</td>
<td>0.716**</td>
</tr>
<tr>
<td>Match-Spec.</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Contest-Spec.</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Time Effects</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Contest-Spec.</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Trend</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

$$N = 1647$$  $$F(9, 532) = 7.66$$  $$N = 1647$$  $$F(10, 532) = 7.53$$  $$N = 1642$$  $$F(14, 531) = 6.45$$  $$N = 649$$  $$F(14, 397) = 3.99$$

Note: dependent variable – individual effort measured as $\tilde{e}_{ij} = \hat{u} - \tilde{u}_{ij}$ where $\tilde{u}_{ij}$ is a number of unforced errors per set player $i$ made in match $j$, $\hat{u} = \max_i \tilde{u}_{ij}$; standard errors are clustered over players; *, **, and *** correspond to 90%, 95%, and 99% significance; other controls include seeding policies of the contests, players’ and their competitors’ individual characteristics, betting odds, growth rates of $A_{t}$ and $\Delta_{A,t}$, total prize money; specification (4) includes only contestants holding at most 1,000 ranking points and playing in matches with $het_{ij} > 300$. 

66
Table 17: Number of Unforced Errors and Risk-Taking Behavior

<table>
<thead>
<tr>
<th></th>
<th>Model I</th>
<th>Model II</th>
<th>Model III</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fastest Serve</td>
<td>0.03</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Speed, km/h</td>
<td>(0.04)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average 1st Serve Speed, km/h</td>
<td>No</td>
<td>–0.9E–03</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.035)</td>
<td></td>
</tr>
<tr>
<td>Average 2nd Serve Speed, km/h</td>
<td>No</td>
<td>No</td>
<td>0.02</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.03)</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
N &= 813 \\
F(12, 367) &= 3.3
\end{align*}
\]

Note: dependent variable – number of unforced errors a player made per set; standard errors are clustered over players; *, **, and *** correspond to 90%, 95%, and 99% significance; other controls include time controls, contest-specific trends, seeding policies, players’ and their competitors’ individual characteristics.
Appendix C. Structural Modeling: Assumptions on the Player-Specific Noise Distribution

In the structural setup, I assume that contestants get additional individual-specific utility (or disutility) when lose:

\[ U_i^L (A^L, B^L, \varepsilon_i) = c_i (\alpha_i A^L + \beta_i B^L) + \varepsilon_i, \varepsilon_i \sim iid F_i (\varepsilon), \varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}] \]

where \( F_i (\varepsilon) \) is a truncation of a mean zero normal distribution with standard deviation \( \sigma_\varepsilon \):

\[ f_i (\varepsilon) = \frac{1}{\sigma_\varepsilon \Phi \left( \frac{\varepsilon}{\sigma_\varepsilon} \right)} \phi \left( \frac{\varepsilon}{\sigma_\varepsilon} \right), \varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}] \]

With this approach, contestants’ types become random:

\[ \tilde{t}_i = c_i \alpha_i \Delta A + c_i \beta_i \Delta B - \varepsilon_i = t_i - \varepsilon_i \]

To match the original theoretical model, I must restrict the support of \( \varepsilon_i \) in the following way:

**Assumption 1:** Losing never results in negative utility:

\[ U_i^L (A^L, B^L, \varepsilon_i) = c_i (\alpha_i A^L + \beta_i B^L) + \varepsilon_i \geq 0 \Leftrightarrow \underline{\varepsilon}_i = -c_i (\alpha_i A^L + \beta_i B^L) \]

Further, recall the definition of contestants’ types. Player \( i \) is the “greediest-to-win” (\( \tilde{t}_i = \tilde{t}^g \)) if and only if \( \tilde{t}_i > \tilde{t}_{-i} \):

\[ \tilde{t}_i > \tilde{t}_{-i} \Leftrightarrow \varepsilon_i - \varepsilon_{-i} < t_i - t_{-i} \]

Now, the identities refer to a particular realization of the preference shocks. In the original model, players randomize uniformly on \( (0, \tilde{t}^n) \) if and only if \( \tilde{t}^n > 0 \) (see Proposition 1). This implies \( e_{ij} \leq \tilde{t}^n_j \) for both types. Then, I impose another constraint on the support of \( f_i (\varepsilon) \):

**Assumption 2:** \( \bar{\varepsilon}_i = t^g_j - \max \{ e_{nj}, e_{gj} \}, i = \{ g, n \} \) for every match \( j \).

Assumption 2 states that the noise distribution for type \( g \) depends on his opponent’s characteristics. Moreover, the condition implies \( \tilde{t}^g_j > 0 \) for any match \( j \) and, as a consequence, the non-trivial equilibrium.

Finally, the support of the noise distribution must be well-defined. I introduce Assumption 3 to guarantee \( \bar{\varepsilon}_i > \underline{\varepsilon}_i \):

**Assumption 3:** \( \max \{ e_{nj}, e_{gj} \} < \min \{ U^W_{n,j}, U^W_{g,j} \} \) for every match \( j \).
Appendix D. Estimation Results and the Goodness-of-Fit: Relative Effort

I checks if the structural model replicates the key empirical patterns in effort. Fitting this variable in the proposed framework is potentially problematic. If the model is a good approximation of contestants’ behavior, the athletes must play mixed strategies. Then, the sum of observed effort levels can differ from its expected value characterized in the model. For this reason, I compare not absolute but relative effort between various groups of players.

Definition 10. Let there be $k \geq 1$ ordered groups, and $e_k$ denotes average expected effort of group $k$. A chain ratio between groups $k$ and $k-1$ is the ratio of effort levels in these groups:

$$
\rho_{k,k-1} = \frac{e_k}{e_{k-1}}, \text{ and } \rho_{1,0} = 1
$$

First, I compare actual and predicted chain ratios over different age and ranking groups.\(^8\) The former partition looks as follows:

$$
d^\text{age}_k \in D^\text{age}, D^\text{age} = \{\min_i (\text{age}_i), 20, 25, 30, \max_i (\text{age}_i)\}
$$

$$
i \in k \iff \text{age}_i \in [d^\text{age}_k, d^\text{age}_{k+1}], k = \{1, \ldots, 4\}
$$

where $k$ is the group. The partition based on ranking points is given by variable $q$ (see Subsection 4.1). In addition, I trace the total effort in competitions where contestants from two extreme elements of each partition meet.

Further, I divide matches into groups based on contestants’ heterogeneity. To approximate this value in the data, I use two approaches:

1. Take absolute differences in contestants’ ranking points for every match $j$:

$$
\text{het}_j^p = |\text{points}_{ij} - \text{points}_{-ij}|
$$

In the structural model, I treat players’ valuations and skills as functions of their ranking points. As a result, the proposed approach must capture some heterogeneity between players in the data.

2. Take absolute differences in contestants’ betting odds for every match $j$:

\(^8\)For every group, I take the average of expected effort over games and simulations. In the data, I repeat the same procedure for individual effort choices. This determines predicted and empirical chain ratios.
\[ het_j^B = |Bet_{ij} - Bet_{-ij}| \]

Betting odds may reflect not only contestants’ rankings but also additional information about their relative strength.\(^8^4\)

Partitions based on contestants’ heterogeneity are summarized as follows:

\[ a_{k}^{het,l} \in \mathcal{D}_{het,l}, l = \{p, B\} \]
\[ j \in k \Leftrightarrow het_j^l \in \left[ a_{k}^{het,l}, a_{k+1}^{het,l} \right], k = \{1, \ldots, \#\mathcal{D}_{het,l} - 1\} \]

where \(k\) is the group, \(p\) and \(B\) refer to the approaches used to approximate heterogeneity in the data.\(^8^5\)

Figure 12 contains predicted and actual chain ratios. On average, the model replicates the shapes of the empirical curves well although sometimes the magnitude of predicted peaks and falls is not exactly captured.

\(^8^4\) This additional information could be injuries, long recovery, changes in the training schedule etc.

\(^8^5\) The partitions are based on empirical distributions of \(het_j^p\) and \(het_j^B\):

\[ \mathcal{D}_{het,p} = \{0, 200, 500, 1000, 1600, 3000, \max_j (het_j^p)\} \]
\[ \mathcal{D}_{het,B} = \{0, 1, 1.93, 6, 8, 11, \max_j (het_j^B)\} \]

The thresholds were subject to robustness checks.
Figure 12: Goodness–of–Fit: Chain Ratios

Note: the points-based partition is driven by variable $q$ (see Subsection 4.1); other partitions look as follows:

\[
d^\text{age}_k \in D^{\text{age}}, D^{\text{age}} = \{\min_i (\text{age}_i), 20, 25, 30, \max_i (\text{age}_i)\}
\]

\[
i \in k \Leftrightarrow \text{age}_i \in [d^\text{age}_k, d^\text{age}_{k+1}), k = \{1, \ldots, 4\}
\]

\[
d^{\text{het},l}_k \in D^{\text{het},l}, l = \{p, B\}
\]

\[
j \in k \Leftrightarrow \text{het}_j^l \in [d^{\text{het},l}_k, d^{\text{het},l}_{k+1}), k = \{1, \ldots, \#D^{\text{het},l} - 1\}
\]

\[
D^{\text{het},p} = \{0, 200, 500, 1000, 1600, 3000, \max_j (\text{het}^p_j)\}
\]

\[
D^{\text{het},B} = \{0, 1, 1.93, 6, 8, 11, \max_j (\text{het}^B_j)\}
\]

\[
\text{het}_j^p = |\text{points}_{ij} - \text{points}_{-ij}|, \text{het}_j^B = |\text{Bet}_{ij} - \text{Bet}_{-ij}|
\]

where $k$ is the group, $p$ and $B$ refer to the approaches used to approximate heterogeneity in the data; group 5 (10) in subplot “Chain Ratios: Age” (“Chain Ratios: Points”) corresponds to mean expected aggregate effort in the matches where contestants from two extreme elements of the partition, 1 and 4 (1 and 9), meet.
Appendix E. Proofs

Proposition 1. For $t_n \leq 0$ the equilibrium is always trivial. For $t_n > 0$ the equilibrium is unique and non-trivial:

- Contestant $g$ randomizes uniformly on $[0, t_n]$, and his equilibrium payoff is $\pi_g = t_g - t_n + U_g^L \geq U_g^L$.

- Contestant $n$ randomizes uniformly on $(0, t_n]$, places the atom of size $p_n^0 = \frac{t_n - t_n}{t_g}$ at $e = 0$, and his equilibrium payoff is $\pi_n = U_n^L \leq \pi_g$.

Proof. Let $G_i(e)$ be a probability that contestant $i$ chooses at most $e$, and $e_i, \bar{e}_i, i = \{g, n\}$ are lower and upper bounds of $G_i(e)$’s support. Denote $\pi_i^W, \pi_i^L, i = \{g, n\}$ as payoffs player $i$ gets when wins or loses, respectively.

First, I show that non-trivial equilibria cannot be obtained under $t_n \leq 0$. Take contestant $n$ and assume $e_n > 0$ in equilibrium. If $n$ wins, he gets $\pi_n^W(e_n) = t_n + U_n^L - e_n$. In case of losing, the payoff becomes $\pi_n^L(e_n) = U_n^L - e_n$, and $\pi_n^W \leq \pi_n^L \forall e_n > 0$ under $t_n \leq 0$. Since losing is preferred to winning, $e_n > 0$ is dominated by $e_n = 0$, and $e_n > 0$ cannot be an equilibrium. Hence, $G_n(0) = 1$, and there are no non-trivial equilibria.

Second, I show that a non-trivial equilibrium exists under $t_n > 0$ by constructing it. Consider player $n$. Let $e_1^n \neq e_2^n$ be two effort levels from $[e_n, \bar{e}_n]$. In equilibrium $e_1^n, e_2^n$ must generate the same expected payoff:

$$G_g(e_1^n) [U_n^W - e_1^n] + (1 - G_g(e_1^n)) [U_n^L - e_1^n] = G_g(e_2^n) [U_n^W - e_2^n] + (1 - G_g(e_2^n)) [U_n^L - e_2^n]$$

or rearranging terms and using a definition of $t_n$:

$$\frac{G_g(e_1^n) - G_g(e_2^n)}{e_1^n - e_2^n} = \frac{1}{t_n}$$

where the right-hand side is constant and does not depend on players’ actions. Then taking $e_1^n - e_2^n \to 0$ I get:

$$g_g(e) = \frac{1}{t_n}, \quad G_g(e) = \frac{e}{t_n}$$

Next, I analyze player $g$ and consider $e_1^g \neq e_2^g$ from $[e_g, \bar{e}_g]$. The equilibrium requires:

$$G_n(e_1^g) [U_g^W - e_1^g] + (1 - G_n(e_1^g)) [U_g^L - e_1^g] = G_n(e_2^g) [U_g^W - e_2^g] + (1 - G_n(e_2^g)) [U_g^L - e_2^g]$$

and after simplifications:

$$\frac{G_n(e_1^g) - G_n(e_2^g)}{e_1^g - e_2^g} = \frac{1}{t_g}$$
Again, taking $e^1_n - e^2_n \to 0$ delivers $g_n(e) = \frac{1}{t_g}$ and $G_n(e) = \frac{e}{t_g}$. Since both density functions $G_g(e)$ and $G_n(e)$ are continuous, there are no atom points within $[\bar{e}_g, \bar{e}_g]$ and $[\bar{e}_n, \bar{e}_n]$.

Further I characterize supports of contestants’ strategies:

**Lemma 1.** $\bar{e}_g = \bar{e}_n = t_n$ in equilibrium.

**Proof.** The proof proceeds in three steps:

1. $\bar{e}_i \leq t_n, i = \{g, n\}$.
   
   Take player $n$ and suppose he chooses $e_n = t_n + \varepsilon_n, \varepsilon_n > 0$ is small enough. Then it must be:
   
   $\pi_n^W(t_n + \varepsilon_n) = U_n^L - \varepsilon_n < \pi_n^W(t_n) = U_n^L$
   
   and $e_n = t_n$ dominates $e_n = t_n + \varepsilon_n$. Hence, $e_n > t_n$ is never played in equilibrium.
   
   Now consider player $g$ and assume $e_g = t_n + \varepsilon_g \in (t_n, t_g), \varepsilon_g > 0$ is small enough. As $n$ never bids above $t_n$, $g$ wins with certainty and gets $\pi_g^W(t_n) = t_g - t_n - \varepsilon_g + U_g^L > 0$. If $g$ chooses $e_g = t_n$, he also succeeds with probability 1, but bears lower costs:
   
   $\pi_g^W(t_n) = t_g - t_n + U_g^L$
   
   Hence, $e_g = t_n$ dominates $e_g = t_n + \varepsilon_g$ for any $\varepsilon_g > 0$, and $\bar{e}_g \leq t_n$ follows.

2. $\bar{e}_n = \bar{e}_g$.
   
   Suppose $\bar{e}_n > \bar{e}_g$ and take player $n$. The contestant wins with certainty when bids $\bar{e}_n \in (\bar{e}_g, \bar{e}_n]$. However, $e_n = \bar{e}_g$ also results in player $n$’s success and dominates $e_n \in (\bar{e}_g, \bar{e}_n]$:
   
   $\pi_n^W(\bar{e}_n) = t_n + U_n^L - \bar{e}_n < \pi_n^W(\bar{e}_g) = t_n + U_n^L - \bar{e}_g \forall \bar{e}_n \in (\bar{e}_g, \bar{e}_n]$
   
   As a result, $\bar{e}_n > \bar{e}_g$ cannot hold in equilibrium. Similar arguments apply to player $g$’s behavior when $\bar{e}_n < \bar{e}_g$ is assumed.

3. $\bar{e}_n = \bar{e}_g = t_n$.
   
   Assume $\bar{e}_n = \bar{e}_g = k < t_n$ and consider contestant $g$. If the player deviates towards $e_g \in (k, t_n]$, he wins with certainty and generates no losses in terms of expected utility. The same is true for contestant $n$.
   
   As a result, $\bar{e}_n = \bar{e}_g = t_n$ must hold in equilibrium. 

\[\square\]
Lemma 2. \( \xi_g = \xi_n = 0 \) in equilibrium.

Proof. \( \xi_i \geq 0, i = \{g, n\} \) follows from a feasibility constraint imposed on \( \xi_i \). Next, I prove two statements:

1. \( \xi_g = \xi_n \).

Assume \( \xi_g > \xi_n \) and consider a best response of player \( g \) to \( \xi_n \in (\xi_n, \xi_g - \varepsilon), \varepsilon \in (0, \xi_g - \xi_n) \).

When bidding \( \xi_g = \xi_g \), contestant \( g \) wins with certainty and gets \( \pi_g^W (\xi_g) = t_g + U_g^L - \xi_g \).

A deviation towards \( \hat{\xi}_g = \xi_g - \eta, \eta \in (0, \varepsilon) \) also results in \( g \)'s success, but leads to higher payoff:

\[
\pi_g^W (\hat{\xi}_g) = t_g + U_g^L - \xi_g + \eta > \pi_g^W (\xi_g) = t_g + U_g^L - \xi_g \quad \forall \eta \in (0, \varepsilon)
\]

Hence, under \( \xi_g > \xi_n \) there exists a profitable deviation contestant \( g \) can implement, and this cannot be an equilibrium. Similar arguments apply to player \( n \)'s behavior when \( \xi_g < \xi_n \) is assumed.

2. \( \xi_n = 0 \). Suppose \( \xi_n = l > 0 \). If contestant \( n \) loses with \( \xi_n = \xi_n \), he gets \( \pi_n^L (\xi_n) = U_n^L - \xi_n < \pi_n^L (0) \). As a result, the player can profitably deviate towards \( \xi_n = 0 \), and this will constitute an equilibrium.

Two results combined together imply \( \xi_g = \xi_n = 0 \) in equilibrium.

Now I can provide a complete characterization of contestants' strategies:

- \( G_g (\hat{\xi}_g) = 1 \) and \( G_g (\xi_g) = 0 \) imply that \( g \)'s strategy has no atoms, and the player randomizes continuously in \([\xi_g, \hat{\xi}_g]\) with a corresponding expected effort \( \frac{t_g^2}{2l_g} \);

- \( G_n (\hat{\xi}_n) = \frac{t_n}{l_n} < 1 \) signals that \( n \)'s strategy contains an atom at zero, and it must be \( G_n (\xi_n) = \frac{t_g - t_n}{l_g} \). Summing up, player \( n \) uses continuous randomization on \([0, t_n]\), places \( p_n^0 = \frac{t_n}{l_n} \) at 0, and his expected effort is \( \frac{t_n^2}{2l_n} \);

- Equilibrium payoffs are \( \pi_g = t_g - t_n + U_g^L \) and \( \pi_n = U_n^L \), and it must be \( \pi_g \geq \pi_n \).

Lemma 3. \( t_g \geq t_n \) implies \( \pi_g \geq \pi_n \).

Proof. The inequality \( \pi_g \geq \pi_n \) can be rewritten in the following way:

\[
\pi_g \geq \pi_n \iff t_g - t_n \geq U_n^L - U_g^L \iff U_g^W - U_n^W \geq 0
\]

Several possible outcomes emerge:

1. \( U_n^L \leq U_g^L \Rightarrow \pi_g \geq \pi_n \) follows from \( t_g \geq t_n \) by definition of contestant types;
2. $U_n^L > U_g^L$:

(a) $U_g^W > U_n^W \Rightarrow t_g \geq t_n$ implies $\pi_g \geq \pi_n$;

(b) $U_g^W < U_n^W \Rightarrow$ it must be:

$$\begin{cases}
U_g^W < U_n^W \\
U_g^L < U_n^L
\end{cases}$$

Summing these conditions up results in $U_g^W + U_g^L < U_n^W + U_n^L$, or equivalently, $U_g^W - U_n^W < U_n^L - U_g^L$. Combining this constraint with $t_g > t_n$ delivers

$$\begin{cases}
U_g^W - U_n^W < U_n^L - U_g^L \\
U_g^W - U_n^W > U_g^L - U_n^L
\end{cases}$$

and a set of feasible $\{U_g^W - U_n^W\}$ is non-empty $\iff U_n^L < U_g^L$, a contradiction.

As a result, $t_g \geq t_n$ implies $\pi_g \geq \pi_n$, and player $g$ never gets less than his competitor $n$.

Since there is no any other candidate equilibrium, uniqueness follows by construction.
Proposition 2. For any $\hat{\Delta}_B \in [-1, 1]$ and valuation profile $r \in R$ the designer

- Uses both goods and leaves a positive losing prize in dimension $A$, or
- Uses both goods and gives the endowment of $A$ to a winner, or
- Does not use dimension $B$ and gives the endowment of $A$ to a winner.

Proof. Let $r = \{\alpha_g, \alpha_n, \beta_g, \beta_n\}$. A complete characterization of the designer’s problem under fixed $B^W, B^L$ looks as follows:

$$
\max_{A_W, A_L} \left\{ J \left( A^W, A^L \right) + \eta_W A^W + \eta_L A^L + \lambda \left( 1 - A^W - A^L \right) \right\} \quad (1)
$$

Abusing notations, re-define $J \left( A^W, A^L, \hat{B}^W, \hat{B}^L \right) \equiv J \left( \Delta_A, \hat{\Delta}_B \right)$. Let $g = i, i = \{1, 2\}$ if and only if $t_i \left( 1, \hat{\Delta}_B \right) > t_{i-1} \left( 1, \hat{\Delta}_B \right)$, which implies

$$
\alpha_g + \beta_g \hat{\Delta}_B > \alpha_n + \beta_n \hat{\Delta}_B \iff \alpha_g > \alpha_n + (\beta_n - \beta_g) \hat{\Delta}_B \quad (2)
$$

Assume no good is left out.

Lemma 1. Problem (1) has no interior solution.

Proof. Consider a second-order derivative of $J \left( \Delta_A, \hat{\Delta}_B \right)$:

$$
\frac{\partial^2 J}{\partial \Delta_A^2} \left( \Delta_A, \hat{\Delta}_B \right) = \frac{\Delta_A^2 (\beta_n \alpha_g - \beta_g \alpha_n)^2}{t_g^2} > 0
$$

Then, any $\Delta^*_A$ such that $\frac{\partial J}{\partial \Delta_A} \left( \Delta^*_A, \hat{\Delta}_B \right) = 0$ corresponds to an interior minimum of $J \left( \Delta_A, \hat{\Delta}_B \right)$.

Corollary 1. $\frac{\partial J}{\partial \Delta_A} \left( \Delta_A, \hat{\Delta}_B \right)$ strictly increases in $\Delta_A$.

Let $\hat{\Delta}_A$ be a prize spread in dimension $A$ such that contestants’ types, $t_n$ and $t_g$, equalize given $\hat{\Delta}_B$:

$$
t_g \left( \Delta_A, \hat{\Delta}_B \right) = t_n \left( \Delta_A, \hat{\Delta}_B \right) \iff \Delta_A = \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n} \hat{\Delta}_B
$$

One can show that $\Delta_A$ is never feasible for $\alpha_g < \alpha_n$:

$$
\Delta_A \leq 1 \iff \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n} \hat{\Delta}_B \leq 1 \iff \alpha_g \leq \alpha_n + (\beta_n - \beta_g) \hat{\Delta}_B
$$

where the latter inequality contradicts condition 2. Hence, a necessary condition for $\Delta_A = \tilde{\Delta}_A$ being feasible is $\alpha_g > \alpha_n$. Further, we will work only with this restriction.
Lemma 2. $\frac{\partial J}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B)$ has a jump point at $\Delta_A = \tilde{\Delta}_A$ for any $\alpha_g \neq \alpha_n$.

Proof. To prove the statement, compute the limits of $\frac{\partial J}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B)$ when it approaches $\Delta_A = \tilde{\Delta}_A$ from the left and from the right:

$$\lim_{\Delta_A \to -\Delta_A} \frac{\partial J}{\partial \Delta_A} (\cdot) = \frac{3\alpha_g - \alpha_n}{2}$$
$$\lim_{\Delta_A \to +\Delta_A} \frac{\partial J}{\partial \Delta_A} (\cdot) = \frac{3\alpha_n - \alpha_g}{2}$$

The limits are equal if and only if $\alpha_g = \alpha_n$. Thus, $\frac{\partial J}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B)$ has a jump point at $\Delta_A = \tilde{\Delta}_A$ for any $\alpha_g \neq \alpha_n$.

Assume $\frac{\partial J}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B)$ is a right-continuous function. Here and after, $\frac{\partial J}{\partial \Delta_A} - (\tilde{\Delta}_A, \hat{\Delta}_B)$ and $\frac{\partial J}{\partial \Delta_A} + (\tilde{\Delta}_A, \hat{\Delta}_B)$ denote derivatives of $J (\cdot)$ to the left and to the right of $\Delta_A = \tilde{\Delta}_A$, respectively.

Given $\Delta_B$, contestants’ types are well-defined if and only if

$$t_j (\Delta_A, \hat{\Delta}_B) \geq 0 \iff \Delta_A^j \geq -\frac{\hat{\Delta}_B j}{\alpha_j}, j = \{g, n\}$$

Depending on players’ preferences, $\Delta_A^j$ can be to the left or to the right of $\tilde{\Delta}_A$.

**Definition.** $\tilde{\Delta}_A$ is type-feasible if and only if $\max \{\Delta_A^g, \Delta_A^n\} < \tilde{\Delta}_A$.

**Lemma 3.** For any valuation profile such that $\tilde{\Delta}_A$ is not type-feasible it is optimal to give the endowment of good $A$ to a winner ($\Delta_A = 1$).

Proof. First, we show that no type-feasibility holds if and only if $\Delta_A^n \geq \tilde{\Delta}_A$. Expanding $\Delta_A^n \geq \tilde{\Delta}_A$ delivers:

$$\Delta_A^n \geq \tilde{\Delta}_A \iff \begin{cases} \frac{\beta_g}{\alpha_g} > \frac{\beta_n}{\alpha_n} \text{ if } \hat{\Delta}_B \geq 0 \\ \frac{\beta_g}{\alpha_g} < \frac{\beta_n}{\alpha_n} \text{ if } \hat{\Delta}_B < 0 \end{cases}$$

where conditions imposed on contestants’ marginal rates of substitution correspond to $\max \{\Delta_A^g, \Delta_A^n\} = \Delta_A^n$ for given $\hat{\Delta}_B$. When $\Delta_A = \Delta_A^n$, the designer ends up with zero expected aggregate effort, and this allocation is strictly dominated by $\Delta_A > \Delta_A^n$. Hence, under $\Delta_A^n \geq \tilde{\Delta}_A$, the choice set is bounded by $\Delta_A \in (\Delta_A^n, 1]$.

Second, consider the derivative $\frac{\partial J}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B)$ at $\Delta_A = \Delta_A^n$:

$$\frac{\partial J}{\partial \Delta_A} (\Delta_A^n, \hat{\Delta}_B) = \frac{\alpha_n}{2} > 0$$

---

86Continuity of $\frac{\partial J}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B)$ at any $\Delta_A \neq \tilde{\Delta}_A$ can be proved easily.

87Type-feasibility is introduced to avoid confusions with a standard feasibility notion, which means $\Delta_A \in [0, 1]$. 

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Since \( \frac{\partial J}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B) \) is positive at \( \Delta_A = \Delta^n_A \) and \( \frac{\partial J}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B) \) strictly increases in \( \Delta_A \) (Corollary 1), it must be \( \frac{\partial J}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B) > 0 \) for any \( \Delta_A \in (\Delta^n_A, 1] \). As a result, \( J(\cdot) \) strictly increases in \( \Delta_A \), and \( \Delta_A = 1 \) is optimal.

\[ \square \]

**Lemma 4.** For any valuation profile such that \( \hat{\Delta}_A \) is type-feasible the objective function \( J(\cdot) \) strictly increases in \( \Delta_A \) for \( \Delta_A \in \left[ \Delta^q_A, \hat{\Delta}_A \right) \). \( J(\cdot) \) strictly decreases or is non-monotone in \( \Delta_A \) for \( \Delta_A \in \left[ \hat{\Delta}_A, 1 \right] \) if and only if \( \alpha_g > 3\alpha_n \); otherwise, \( J(\cdot) \) strictly increases in \( \Delta_A \) for \( \Delta_A \in \left[ \hat{\Delta}_A, 1 \right] \).

**Proof.** In the beginning, we prove that type-feasibility holds if and only if \( \Delta^q_A < \hat{\Delta}_A \): 

\[ \Delta^q_A < \hat{\Delta}_A \iff \begin{cases} \frac{\partial g}{\partial \alpha_g} < \frac{\partial n}{\partial \alpha_n} & \text{if } \hat{\Delta}_B \geq 0 \\ \frac{\partial g}{\partial \alpha_g} > \frac{\partial n}{\partial \alpha_n} & \text{if } \hat{\Delta}_B < 0 \end{cases} \]

where latter inequalities guarantee \( \max \{ \Delta^q_A, \Delta^n_A \} = \Delta^q_A \) for any \( \hat{\Delta}_B \). Allocating \( \Delta_A = \Delta^q_A \) results in zero expected aggregate effort, and the designer’s choice set is bounded by \( \Delta_A \in (\Delta^q_A, 1] \).

Since \( \hat{\Delta}_A \) is type-feasible and \( \frac{\partial J}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B) \) has a jump point at \( \Delta_A = \hat{\Delta}_A \), one must analyze the intervals \( \Delta_A \in \left[ \Delta^q_A, \hat{\Delta}_A \right] \) and \( \Delta_A \in \left[ \hat{\Delta}_A, 1 \right] \) separately. Consider \( \Delta_A \in \left[ \Delta^q_A, \hat{\Delta}_A \right] \) and evaluate \( \frac{\partial J}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B) \) at \( \Delta_A = \Delta^q_A \):

\[ \frac{\partial J}{\partial \Delta_A} (\Delta^q_A, \hat{\Delta}_B) = \frac{\alpha_g}{2} > 0 \]

Given \( \frac{\partial J}{\partial \Delta_A} (\Delta^q_A, \hat{\Delta}_B) > 0 \) and Corollary 1, \( J(\cdot) \) must strictly increase in \( \Delta_A \) for any \( \Delta_A \in \left[ \Delta^q_A, \hat{\Delta}_A \right] \).

Now, take \( \Delta_A \in \left[ \hat{\Delta}_A, 1 \right] \) and recall Lemma 2:

\[ \frac{\partial J}{\partial \Delta_A}^+ (\hat{\Delta}_A, \hat{\Delta}_B) = \frac{3\alpha_n - \alpha_g}{2} \]

and \( \frac{\partial J}{\partial \Delta_A} (\cdot, \hat{\Delta}_B) \geq 0 \) if and only if \( \alpha_g \leq 3\alpha_n \). Given Corollary 1, under \( \alpha_g \leq 3\alpha_n \) and \( \frac{\partial J}{\partial \Delta_A} (1, \hat{\Delta}_B) > 0 \) there exists a unique value \( \Delta^*_A \in \left( \hat{\Delta}_A, 1 \right) \) such that \( \frac{\partial J}{\partial \Delta_A} (\Delta^*_A, \hat{\Delta}_B) = 0 \) and \( J(\cdot) \) strictly decreases (increases) in \( \Delta_A \) for \( \Delta_A \in \left( \hat{\Delta}_A, \Delta^*_A \right) \) (\( \Delta_A \in \left( \Delta^*_A, 1 \right) \)). When \( \alpha_g \leq 3\alpha_n \) and \( \frac{\partial J}{\partial \Delta_A} (1, \hat{\Delta}_B) < 0 \), the objective function \( J(\cdot) \) strictly decreases in \( \Delta_A \) for \( \Delta_A \in \left( \hat{\Delta}_A, 1 \right] \).

\[ \square \]

To sum up, type-feasibility requires:
\[
\begin{aligned}
    \alpha_g &> \max \left\{ \alpha_n + (\beta_n - \beta_g) \hat{\Delta}_B, \alpha_n, \alpha_n \beta_n \right\} \quad \text{if } \hat{\Delta}_B \geq 0 \\
    \alpha_g &\in \left( \alpha_n + (\beta_n - \beta_g) \hat{\Delta}_B, \alpha_n \beta_n \right) \quad \text{if } \hat{\Delta}_B < 0 \\
    \beta_g &> \beta_n
\end{aligned}
\] (3)

Condition \( \alpha_g > 3\alpha_n \) is necessary (but not sufficient) to make \( \Delta_A < 1 \) optimal. When it holds and \( \hat{\Delta}_A \) is type-feasible, only two candidate prize allocations with both goods used are left and must be compared directly (Lemma 1 excludes interior solutions):

\[ (1, \hat{\Delta}_B) \text{ or } (\hat{\Delta}_A, \hat{\Delta}_B) \]

The designer cannot change the prize allocation in dimension \( B \) and may prefer to leave this item out.

**Lemma 5.** If dimension \( B \) is left out, it is optimal to give the endowment of good \( A \) to a winner (\( \Delta_A = 1 \)).

*Proof.* When there is no dimension \( B \), \( \frac{\partial J}{\partial \Delta_A} (\Delta_A, 0) \) looks as follows:

\[
\frac{\partial J}{\partial \Delta_A} (\Delta_A, 0) = \frac{\alpha_n (\alpha_n + \alpha_g)}{2\alpha_g} I_{\{\alpha_g > \alpha_n\}} + \frac{\alpha_g (\alpha_g + \alpha_n)}{2\alpha_n} \left[ 1 - I_{\{\alpha_g > \alpha_n\}} \right] > 0
\]

where \( I_{\{\alpha_g > \alpha_n\}} = 1 \) if \( \alpha_g > \alpha_n \) and \( I_{\{\alpha_g > \alpha_n\}} = 0 \), otherwise. Given that \( J(\cdot) \) strictly increases in \( \Delta_A \), it is optimal to choose \( \Delta_A = 1 \) and use good \( A \)'s endowment completely.

\( \square \)

Here and after, a prize schedule where dimension \( B \) is omitted is denoted as \( (1, 0)_{\{\alpha_g > \alpha_n\}} \).

Now, we analyze the cases when type-feasibility holds and the objective function \( J(\cdot) \) strictly decreases or is non-monotone in \( \Delta_A \) for \( \Delta_A \in (\hat{\Delta}_A, 1] \), i.e. \( \alpha_g > 3\alpha_n \):

1. \( \hat{\Delta}_A \leq -1 \), i.e. \( \hat{\Delta}_A \) is not feasible.

   This condition holds if and only if

   \[ \alpha_g \leq \alpha_n - (\beta_n - \beta_g) \hat{\Delta}_B \] (4)

   For \( \hat{\Delta}_B \geq 0 \), (4) always has a non-empty intersection with (3) if and only if

   \[
   \begin{align*}
   \alpha_g &\in \left( 3\alpha_n, \alpha_n - (\beta_n - \beta_g) \hat{\Delta}_B \right) \\
   \alpha_n &\in \left[ 0, \min \left\{ \frac{(\beta_n - \beta_n)}{2}, \beta_n \hat{\Delta}_B \right\} \right) \\
   \beta_g &> \beta_n > 0
   \end{align*}
   \] (5)

   When \( \hat{\Delta}_B < 0 \), (4) combined with (3) always delivers an empty set for \( \beta_g > \beta_n \). Hence, under \( \hat{\Delta}_A \leq -1 \), we study only the case of \( \hat{\Delta}_B \geq 0 \).
Next, one must compare $J(-1, \hat{\Delta}_B)$, $J(1, \hat{\Delta}_B)$, and $J(1, 0)_{\{\alpha_g > \alpha_n\}}$. In the beginning, consider $J(-1, \hat{\Delta}_B)$ and $J(1, \hat{\Delta}_B)$:

$$J(-1, \hat{\Delta}_B) > J(1, \hat{\Delta}_B) \iff \alpha_n \alpha_g^2 + \left(\alpha_n^2 + \beta_n \hat{\Delta}_B^2\right) \alpha_g - \hat{\Delta}_B^2 \alpha_n \beta_g (\beta_g + 2\beta_n) > 0 \quad (6)$$

The corresponding square equation solved for $\alpha_g$ always has two real roots, and condition (6) holds if and only if

$$\alpha_g > r_1 > 0 \quad (7)$$

where

$$r_1 = \frac{-\left(\alpha_n^2 + \beta_n \hat{\Delta}_B^2\right) - \sqrt{\left(\alpha_n^2 + \beta_n \hat{\Delta}_B^2\right)^2 + 4\hat{\Delta}_B^2 \alpha_n^2 \beta_g (\beta_g + 2\beta_n)}}{2\alpha_n}$$

Conditions (5) and (7) define a non-empty set of $\alpha_g$’s if and only if

$$\alpha_n \in [0, q_1) \quad (8)$$

where

$$q_1 = \frac{(\beta_n - 3\beta_g) + \sqrt{(\beta_n + \beta_g) (9\beta_g - 7\beta_n) \hat{\Delta}_B}}{4} < \min \left\{\frac{(\beta_g - \beta_n)}{2} \hat{\Delta}_B, \beta_n \hat{\Delta}_B\right\}$$

Further, compare $J(-1, \hat{\Delta}_B)$ and $J(1, 0)_{\{\alpha_g > \alpha_n\}}$:

$$J(-1, \hat{\Delta}_B) > J(1, 0)_{\{\alpha_g > \alpha_n\}} \iff$$

$$\left(2\alpha_n - \beta_n \hat{\Delta}_B\right) \alpha_g^2 + \left(2\alpha_n - \left(2\alpha_n - \beta_n \hat{\Delta}_B\right)(\beta_n + \beta_n) \hat{\Delta}_B\right) \alpha_g - \alpha_n^2 \beta_g \hat{\Delta}_B > 0 \quad (9)$$

where $\left(2\alpha_n - \beta_n \hat{\Delta}_B\right) < 0$ for any $\alpha_n \in [0, q_1)$. The underlying square equation always has two real roots, $\bar{r}_1$ and $\bar{r}_2$, and $0 < \bar{r}_1 \leq \bar{r}_2$ when $\alpha_n \in [0, q_1)$. Then, the designer prefers $(-1, \hat{\Delta}_B)$ to $(1, 0)_{\{\alpha_g > \alpha_n\}}$ if and only if $\alpha_g \in (\bar{r}_1, \bar{r}_2)$, and this set combined with (5) and (7) must deliver a non-empty intersection.

**Lemma 6.** For any preference structure such that $\hat{\Delta}_A = -1$ is feasible and $J(-1, \hat{\Delta}_B) > J(1, \hat{\Delta}_B)$, the designer never ignores good $B$.

**Proof.** To prove the lemma, $\bar{r}_1 < 3\alpha_n$ and $\bar{r}_2 > \alpha_n - \hat{\Delta}_B (\beta_n - \beta_g)$ for any $\alpha_n \in [0, q_1)$ are sufficient. Consider the $\bar{r}_1 < 3\alpha_n$ condition. Solving the underlying equation $\bar{r}_1 - 3\alpha_n = 0$ with respect to $\alpha_n$ delivers three roots:
\[ k_1 = \frac{\beta_n \hat{\Delta}_B}{2} > q_1 \]
\[ k_{2,3} = \left( \frac{7\beta_n + 15\beta_n + \sqrt{49\beta_n^2 - 78\beta_n \beta_n - 63\beta_n^2}}{48} \right) \hat{\Delta}_B \]

If \( \beta_g \in \left( \beta_n, \frac{39+48\sqrt{2}}{49} \beta_n \right) \), roots \( k_2 \) and \( k_3 \) are complex. Then, \( \tilde{r}_1 - 3\alpha_n < 0 \) for \( \alpha_n \in [0, k_1) \) and \( \tilde{r}_1 - 3\alpha_n \geq 0 \) for \( \alpha_n \geq k_1 \). Imposing constraint (8), we get \( \tilde{r}_1 - 3\alpha_n < 0 \) for any \( \alpha_n \in [0, q_1) \).

When \( \beta_g \geq \frac{(39+48\sqrt{2})}{49} \beta_n \), both \( k_2 \) and \( k_3 \) become real and positive. Moreover:

\[ k_3 > k_1 > q_1 \text{ for any } \beta_g \geq \frac{(39+48\sqrt{2})}{49} \beta_n \]
\[ k_2 > k_1 > q_1 \text{ for } \beta_g \in \left[ \frac{(39+48\sqrt{2})}{49} \beta_n, 3\beta_n \right) \text{ and } k_2 \leq k_1 \text{ for } \beta_g \geq 3\beta_n \]

With \( \beta_g \in \left[ \frac{(39+48\sqrt{2})}{49} \beta_n, 3\beta_n \right) \), the inequality \( \tilde{r}_1 - 3\alpha_n < 0 \) holds for \( \alpha_n \in [0, k_1) \). Combining this with constraint (8) results in \( \tilde{r}_1 - 3\alpha_n < 0 \) for any \( \alpha_n \in [0, q_1) \). Under \( \beta_g \geq 3\beta_n \), \( \tilde{r}_1 - 3\alpha_n < 0 \) \( (\tilde{r}_1 - 3\alpha_n \geq 0) \) holds for \( \alpha_n \in [0, k_2) \) \( (\alpha_n \in [k_2, k_1)) \). We analyze \( q_1 \) and \( k_2 \) as functions of \( \beta_g \). First, compare \( q_1 (3\beta_g) \) and \( k_2 (3\beta_g) \):

\[ q_1 (3\beta_g) = \left( \sqrt{5} - 2 \right) \beta_n \hat{\Delta}_B < \frac{\beta_n \hat{\Delta}_B}{2} = k_2 (3\beta_g) \]

Second, compute derivatives of \( q_1 \) and \( k_2 \) with respect to \( \beta_g \):

\[ \frac{\partial q_1}{\partial \beta_g} = \frac{9\beta_g + \beta_n - 3\sqrt{(\beta_n + \beta_g)(9\beta_n - 7\beta_n)}}{4(\beta_n + \beta_g)(9\beta_n - 7\beta_n)} \hat{\Delta}_B > 0 \]
\[ \frac{\partial k_2}{\partial \beta_g} = \frac{-49\beta_g^2 + 39\beta_n + 7\sqrt{49\beta_g^2 - 78\beta_n \beta_n - 63\beta_n^2}}{48 \sqrt{49\beta_g^2 - 78\beta_n \beta_n - 63\beta_n^2}} \hat{\Delta}_B < 0 \]

Hence, \( q_1 \) \( (k_2) \) strictly increases \( (\text{decreases}) \) in \( \beta_g \) for any \( \beta_g, \beta_n, \hat{\Delta}_B \geq 0 \), and the equation \( k_2 - q_1 = 0 \) must have at most one root in \( \beta_g \geq 3\beta_n \). Finally, take a limit of \( (k_2 - q_1) \) for \( \beta_g \to \infty \):

\[ \lim_{\beta_g \to \infty} (k_2 - q_1) = \frac{2\beta_n \hat{\Delta}_B}{21} > 0 \]

Given strict monotonicity of \( k_1 \) and \( q_1, k_2 \), \( (3\beta_g) > q_1(3\beta_g) \), and \( \lim_{\beta_g \to \infty} (k_1 - q_1) > 0 \), there does not exist \( \tilde{\beta}_g > 3\beta_n \) such that \( \left( k_1 \left( \tilde{\beta}_g \right) - q_1 \left( \tilde{\beta}_g \right) = 0 \right) \). Thus, \( k_2 > q_1 \) for any \( \beta_g \geq 3\beta_n \). Since \( \tilde{r}_1 - 3\alpha_n < 0 \) for any \( \alpha_n \in [0, k_2) \), imposing constraint (8) delivers the first statement of the lemma.
Next, consider the second inequality, \( \tilde{r}_2 > \alpha_n - \Delta_B (\beta_n - \beta_g) \). Solving \( \tilde{r}_2 = \alpha_n - \Delta_B (\beta_n - \beta_g) \) with respect to \( \alpha_n \) delivers four roots:

\[
d_{1,2} = \frac{\Delta_B}{8} \left( 5\beta_n - \beta_g \mp \sqrt{(\beta_g + \beta_n)(9\beta_g - 7\beta_n)} \right)
\]

where \( d_1 < 0 < d_2 < d_3 < d_4 \) and \( q_1 \in (0, d_2) \) for any \( \beta_g \geq 0 \). The inequality \( \tilde{r}_2 > \alpha_n - \Delta_B (\beta_n - \beta_g) \) holds for \( \alpha_n \in [0, d_2) \). Combining this with condition (8), \( \tilde{r}_2 > \alpha_n - \Delta_B (\beta_n - \beta_g) \) for any \( \alpha_n \in [0, q_1) \) comes out.

Thus, feasibility of \( \tilde{\Delta}_A = -1 \) and \( J(-1, \hat{\Delta}_B) > J(1, \hat{\Delta}_B) \) imply \( J(-1, \hat{\Delta}_B) > J(1, 0)_{\{\alpha_g > \alpha_n\}} \).

\[\square\]

To sum up, \( \Delta_A = -1 \) is optimal if and only if

\[
\begin{cases}
\alpha_g \in \left( \max \{r_1, 3\alpha_n\}, \alpha_n - (\beta_n - \beta_g) \hat{\Delta}_B \right) \\
\alpha_n \in [0, q_1) \\
\beta_g > \beta_n > 0
\end{cases}
\]

When \( \alpha_n \in \left( q_1, \min \left\{ \frac{(\beta_g - \beta_n)\hat{\Delta}_B}{2}, \beta_n\hat{\Delta}_B \right\} \right) \), the set \( \left( r_1, \alpha_n - (\beta_n - \beta_g) \hat{\Delta}_B \right) \) becomes empty. In words, the \( (1, \hat{\Delta}_B) \) bundle strictly dominates the \( (-1, \hat{\Delta}_B) \) allocation. To complete the analysis, one must compare \( J(1, \hat{\Delta}_B) \) and \( J(1, 0)_{\{\alpha_g > \alpha_n\}} \) directly:

\[
J(1, \hat{\Delta}_B) > J(1, 0)_{\{\alpha_g > \alpha_n\}} \Leftrightarrow \beta_n\alpha_g^2 + \beta_n \left( 2\alpha_n + \hat{\Delta}_B (\beta_n + \beta_g) \right) \alpha_g - \alpha_n^2\beta_g > 0
\]  

(10)

The underlying square equation always has two real roots, \( \hat{r}_1 \) and \( \hat{r}_2, \hat{r}_2 < 0 < \hat{r}_1 \). Condition (10) holds if and only if \( \alpha_g > \hat{r}_1 \).

**Lemma 7.** For any preference structure such that \( \tilde{\Delta}_A \leq -1 \), type-feasibility holds and \( J(-1, \hat{\Delta}_B) < J(1, \hat{\Delta}_B) \), the designer never leaves good \( B \) out.

**Proof.** To prove the lemma, it is sufficient to show that \( \hat{r}_1 < \alpha_n \) for any \( \alpha_n \in \left[ 0, \beta_n \hat{\Delta}_B \right) \).

The corresponding equation \( \hat{r}_1 - \alpha_n = 0 \) solved for \( \alpha_n \) has two roots:

\[
c_1 = 0, \quad c_2 = \frac{\hat{\Delta}_B \beta_n (\beta_n + \beta_g)}{\beta_g - 3\beta_n}
\]
When $\beta_g < 3\beta_n$, the root $c_2$ is negative, and $\dot{r}_1 - \alpha_n < 0$ holds for any $\alpha_n \geq 0$. If $\beta_g > 3\beta_n$, $c_2$ becomes positive, and $c_2 > \hat{\Delta}_B\beta_n$. Then, the inequality $\dot{r}_1 - \alpha_n < 0$ is satisfied for any $\alpha_n \in [0, c_2)$. Imposing $\alpha_n \in \left[0, \beta_n\hat{\Delta}_B\right)$, we get $\dot{r}_1 < \alpha_n$ for any $\alpha_n \in \left[0, \beta_n\hat{\Delta}_B\right)$. Thus, $\hat{\Delta}_A \leq -1$, type-feasibility and $J(\nabla, \hat{\Delta}_B) < J(1, \hat{\Delta}_B)$ imply $J(1, \hat{\Delta}_B) > J(1, 0)\left\{\alpha_g > \alpha_n\right\}$.

2. $\hat{\Delta}_A \in (-1, 1)$, i.e. $\hat{\Delta}_A$ is feasible.

This is the case if and only if

$$\alpha_g > \alpha_n - (\beta_n - \beta_g)\hat{\Delta}_B$$

(11)

Condition (11) has a non-empty intersection with (3) and $\alpha_g > 3\alpha_n$ if and only if

$$\begin{cases} 
\alpha_g > \max\left\{\alpha_n + \hat{\Delta}_B\max\left\{\beta_n - \beta_g, \beta_g - \beta_n\right\}, 3\alpha_n, \alpha_n\frac{\beta_n}{\beta_m}\right\} & \text{if } \hat{\Delta}_B \geq 0 \\
\alpha_g \in \left(\max\left\{\alpha_n + (\beta_n - \beta_g)\hat{\Delta}_B, 3\alpha_n\right\}, \alpha_n\frac{\beta_n}{\beta_m}\right) & \text{if } \hat{\Delta}_B < 0 \\
\alpha_g > -\beta_n\hat{\Delta}_B & \text{if } \hat{\Delta}_B < 0 \\
\beta_g > 3\beta_n
\end{cases}$$

(12)

In the beginning, consider the case of $\hat{\Delta}_B \geq 0$. The designer prefers a bundle with $\Delta_A = \hat{\Delta}_A \in (-1, 1)$ to the $(1, \hat{\Delta}_B)$ allocation if and only if

$$J(\hat{\Delta}_A, \hat{\Delta}_B) > J(1, \hat{\Delta}_B) \iff (\beta_n\hat{\Delta}_B - \alpha_n)\alpha_g^2 + \hat{\Delta}_Bl_1\alpha_g + \alpha_nl_2 > 0$$

(13)

where

$$l_1 = \beta_n\hat{\Delta}_B(\beta_g - \beta_n) - \alpha_n(\beta_n + 3\beta_g)$$

$$l_2 = \alpha_n(\alpha_n + \beta_g\hat{\Delta}_B) + 2\hat{\Delta}_B(\alpha_n\beta_n - \beta_g^2) + \hat{\Delta}_B^2\beta_n(\beta_n + \beta_g)$$

Lemma 8. For $\alpha_n \in \left(0, \beta_n\hat{\Delta}_B\right)$ and $\hat{\Delta}_B \geq 0$ there exists $\alpha_g^*$ such that $J(\hat{\Delta}_A, \hat{\Delta}_B) > J(1, \hat{\Delta}_B)$ for any $\alpha_g > \alpha_g^*$.

Proof. The equation corresponding to condition (13) always has two real roots:

$$s_1 = \alpha_n + (\beta_n - \beta_g)\hat{\Delta}_B, \ s_2 = \frac{\alpha_n(\alpha_n + 2\beta_g\hat{\Delta}_B + \beta_n\hat{\Delta}_B)}{\beta_n\hat{\Delta}_B - \alpha_n}$$
where $s_1$ coincides with the lower bound of set (2). First, take $\alpha_n \in (0, \beta_n \hat{\Delta}_B)$ when both $s_1$ and $s_2$ are positive. If $\beta_g \geq \beta_n$, it is always the case that $s_2 > s_1$, and inequality (13) holds if and only if

$$\alpha_g \in [0, s_1) \cup (s_2, \infty)$$

Combining this with condition (12) for $\hat{\Delta}_B \geq 0$, define $\alpha^*_g = \max \{\alpha_n + \hat{\Delta}_B (\beta_g - \beta_n), 3\alpha_n, \alpha_n \beta_n / \beta_g, s_2\}$. Given $\alpha_n \in (0, \beta_n \hat{\Delta}_B)$ and inequality (13), it must be $J(\hat{\Delta}_A, \hat{\Delta}_B) > J(1, \hat{\Delta}_B)$ for any $\alpha_g > \alpha^*_g$.

Next, consider $\beta_g < \beta_n$. If $\alpha_n \in \left(0, \frac{-\beta_n - \beta_g}{2} \hat{\Delta}_B\right]$, it must be $s_2 < s_1$, and condition (13) implies $J(\hat{\Delta}_A, \hat{\Delta}_B) > J(1, \hat{\Delta}_B)$. When $\alpha_n \in \left(\frac{-\beta_n - \beta_g}{2} \hat{\Delta}_B, \beta_n \hat{\Delta}_B\right)$, the outcome is exactly the same as in case of $\beta_g \geq \beta_n$. Thus, choosing $\alpha^*_g = \max \{\alpha_n + \hat{\Delta}_B (\beta_g - \beta_n), 3\alpha_n, \alpha_n \beta_n / \beta_g, s_2, s_1\}$ for $\alpha_n \in (0, \beta_n \hat{\Delta}_B)$, we get the statement of the lemma.

Finally, take $\alpha_n > \beta_n \hat{\Delta}_B$, which results in $s_2 < 0$. Then, condition (13) holds if and only if $\alpha_g \in [0, s_1)$, and this contradicts (12). Hence, the $(\hat{\Delta}_A, \hat{\Delta}_B)$ bundle is never preferred to the $(1, \hat{\Delta}_B)$ allocation when $\alpha_n > \beta_n \hat{\Delta}_B$ and $\hat{\Delta}_B \geq 0$.

\[ \square \]

**Lemma 9.** For any preference structure such that $J(\hat{\Delta}_A, \hat{\Delta}_B) > J(1, \hat{\Delta}_B)$ and $\hat{\Delta}_B \geq 0$ the designer never leaves good $B$ out.

**Proof.** The designer prefers the $(\hat{\Delta}_A, \hat{\Delta}_B)$ bundle to the $(1, 0)_{\{\alpha_g > \alpha_n\}}$ allocation if and only if

$$J(\hat{\Delta}_A, \hat{\Delta}_B) > J(1, 0)_{\{\alpha_g > \alpha_n\}} \Leftrightarrow (2\beta_n \hat{\Delta}_B - \alpha_n) \alpha^2_g - \beta_g \alpha_n \hat{\Delta}_B \alpha_g + \alpha^3_n > 0 \quad (14)$$

where $\mu_1 = 2\beta_n \hat{\Delta}_B - \alpha_n > 0$ for $\alpha_n \in \left(0, \beta_n \hat{\Delta}_B\right)$ and $\hat{\Delta}_B \geq 0$. Two cases emerge:

- $\beta_g \geq \beta_n \Rightarrow$ inequality (14) has two real roots, $\bar{r}_1$ and $\bar{r}_2$, where $\bar{r}_1 < \bar{r}_2$ and $\bar{r}_1 > 0$ for $\alpha_n \in \left(0, \beta_n \hat{\Delta}_B\right)$. Then, the designer prefers $(\hat{\Delta}_A, \hat{\Delta}_B)$ to $(1, 0)_{\{\alpha_g > \alpha_n\}}$ if and only if $\alpha_g \in \left[0, \bar{r}_1\right) \cup (\bar{r}_2, \infty)$. Now, consider how $\bar{r}_2$ relates to $s_2$ defined in Lemma 8. Solving $(\bar{r}_2 - s_2 = 0)$ with respect to $\alpha_n$ delivers two roots:

$$\bar{r}_2 - s_2 = 0 \Leftrightarrow \alpha_n^{1,2} = \hat{\Delta}_B \left(\beta_n \pm \sqrt{2\beta_n (\beta_g + \beta_n)}\right)$$

where $\alpha_n^1 > 0 > \alpha_n^2$ for $\beta_g \geq \beta_n$ and $\alpha_n^1 > \beta_n \hat{\Delta}_B$. The inequality $\bar{r}_2 < s_2$ holds for any $\alpha_n \in [0, \alpha_n^1)$; given $\alpha_n^1 > \beta_n \hat{\Delta}_B$, it must be $\bar{r}_2 < s_2$ under $\alpha_n \in \left(0, \beta_n \hat{\Delta}_B\right)$. Hence,
\[ \alpha_g > s_2 \text{ implies } \alpha_g > \bar{r}_2, \] i.e. for any preference structure such that \( \beta_g \geq \beta_n \) and 

\[ J \left( \hat{\Delta}_A, \hat{\Delta}_B \right) > J \left( 1, \hat{\Delta}_B \right) \] 

hold the designer prefers to use both goods.

- \( \beta_g < \beta_n \Rightarrow \) roots of condition (14) are real if and only if 

\[ \alpha_n \in \left( 0, \hat{\Delta}_B \left( \beta_n - \sqrt{\beta_n^2 - \beta_g^2} \right) \right) \cup \left( \hat{\Delta}_B \left( \beta_n + \sqrt{\beta_n^2 - \beta_g^2} \right), \infty \right) \quad (15) \]

Otherwise, \( J \left( \hat{\Delta}_A, \hat{\Delta}_B \right) > J \left( 1, 0 \right)_{\{\alpha_g > \alpha_n\}} \) holds for any \( \alpha_g \). Suppose (15) is satisfied.

Consider how \( \bar{r}_2 \) and \( (\alpha_n + \hat{\Delta}_B (\beta_n - \beta_g)) \), a lower bound of \( \alpha_g \) from condition (12), locate relative to each other. The equation \( \{ \bar{r}_2 = \alpha_n + \hat{\Delta}_B (\beta_n - \beta_g) \} \) has two roots:

\[ \alpha_n = \frac{2\beta_n \hat{\Delta}_B (\beta_n - \beta_g)}{\beta_g - 3\beta_n} < 0 \quad \text{and} \quad \alpha_n = 2\beta_n \hat{\Delta}_B > \beta_n \hat{\Delta}_B - \text{ and } \bar{r}_2 < \alpha_n + \hat{\Delta}_B (\beta_n - \beta_g) \]

for any \( \alpha_n \in \left[ 0, 2\beta_n \hat{\Delta}_B \right) \). Hence, \( J \left( \hat{\Delta}_A, \hat{\Delta}_B \right) > J \left( 1, \hat{\Delta}_B \right) \) implies \( J \left( \hat{\Delta}_A, \hat{\Delta}_B \right) > J \left( 1, 0 \right)_{\{\alpha_g > \alpha_n\}} \) for \( \alpha_n \in \left( 0, \beta_n \hat{\Delta}_B \right) \), \( \beta_g < \beta_n \), and \( \hat{\Delta}_B \geq 0 \).

\[ \square \]

Under conditions specified in (12) and Lemma 9, \( \Delta_A = \hat{\Delta}_A \) is an optimal prize spread. Then, one must check if the designer uses good A’s endowment completely.

**Lemma 10.** For \( \hat{\Delta}_B \geq 0, \hat{\Delta}_A \in (-1, 1) \) and \( J \left( \hat{\Delta}_A, \hat{\Delta}_B \right) > J \left( 1, \hat{\Delta}_B \right) \) the designer assigns \( A^W, A^L > 0 \) in a reward bundle and does not waste good A’s endowment.

**Proof.** To prove the statement, recall properties of the designer’s objective function established in Lemma 4. When \( J \left( \hat{\Delta}_A, \hat{\Delta}_B \right) > J \left( 1, \hat{\Delta}_B \right) \), it must be

\[ \frac{\partial J^+}{\partial \Delta_A} \left( \hat{\Delta}_A, \hat{\Delta}_B \right) > 0 \]

First-order conditions of the designer’s problem look as follows:

\[ \frac{\partial L}{\partial A^W} \left( \cdot \right) = \frac{\partial L}{\partial \Delta_A} \left( \Delta_A, \hat{\Delta}_B \right) + \eta_W - \lambda \]

\[ \frac{\partial L}{\partial A^L} \left( \cdot \right) = -\frac{\partial L}{\partial \Delta_A} \left( \Delta_A, \hat{\Delta}_B \right) + \eta_L - \lambda \]

Take the case of \( \beta_g \geq \beta_n \) where \( \hat{\Delta}_A \in (-1, 0) \). Suppose \( \lambda = \eta_L = 0, \eta_W > 0 \) must hold in the optimum, i.e. the designer wastes a share of good A’s endowment. Then, it has to be:

\[ \frac{\partial L^+}{\partial \Delta_A} \left( \hat{\Delta}_A, \hat{\Delta}_B \right) = \frac{\partial L}{\partial \Delta_A} \left( \Delta_A, \hat{\Delta}_B \right) + \eta_W = 0 \iff \eta_W = -\frac{\partial L^+}{\partial \Delta_A} \left( \hat{\Delta}_A, \hat{\Delta}_B \right) \]

\[ \frac{\partial L^+}{\partial \Delta_A} \left( \hat{\Delta}_A, \hat{\Delta}_B \right) = -\frac{\partial L}{\partial \Delta_A} \left( \hat{\Delta}_A, \hat{\Delta}_B \right) > 0 \]

\[ \frac{\partial L}{\partial A^W} \left( \Delta_A, \hat{\Delta}_B \right) = \frac{\partial L}{\partial \Delta_A} \left( \Delta_A, \hat{\Delta}_B \right) + \eta_W > 0 \forall \Delta_A \in \left[ \max \left[ \Delta_A^g, -1 \right], \hat{\Delta}_A \right) \]

\[ \frac{\partial L}{\partial A^L} \left( \Delta_A, \hat{\Delta}_B \right) = -\frac{\partial L}{\partial \Delta_A} \left( \Delta_A, \hat{\Delta}_B \right) < 0 \forall \Delta_A \in \left[ \max \left[ \Delta_A^g, -1 \right], \hat{\Delta}_A \right) \]
\( \frac{\partial L}{\partial A^W} (\Delta_A, \hat{\Delta}_B) > 0 \) implies non-optimality of \( A^W = 0 \), and the initial guess \( \eta_W > 0 \) was incorrect. Further, assume \( \lambda > 0 \), \( \eta_{L,W} = 0 \), and first-order conditions become:

\[
\begin{align*}
\frac{\partial L^+}{\partial A^W} (\Delta_A, \hat{\Delta}_B) &= -\frac{\partial J^+}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B) - \lambda = 0 \iff \lambda = -\frac{\partial J^+}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B) \\
\frac{\partial L^+}{\partial A^W} (\Delta_A, \hat{\Delta}_B) &= \frac{\partial J^+}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B) < 0 \\
\frac{\partial L^-}{\partial A^W} (\Delta_A, \hat{\Delta}_B) &= \frac{\partial J^-}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B) - \lambda < 0 \forall \Delta_A \in \left[ \max [\Delta_A^g, -1], \hat{\Delta}_A \right] \\
\frac{\partial L^-}{\partial A^W} (\Delta_A, \hat{\Delta}_B) &= -\frac{\partial J^-}{\partial \Delta_A} (\Delta_A, \hat{\Delta}_B) < 0 \forall \Delta_A \in \left[ \max [\Delta_A^g, -1], \hat{\Delta}_A \right]
\end{align*}
\]

These conditions do not result in a contradiction. For \( \Delta_A \in \left[ \max [\Delta_A^g, -1], \hat{\Delta}_A \right] \) the designer benefits when assigns the highest (the lowest) feasible \( A^W > 0 \) \((A^L > 0)\). Under \( \Delta_A \in \left[ \hat{\Delta}_A, 1 \right] \) it is optimal to choose the smallest prize spread such that both reward components are positive and the endowment is not wasted. Non-optimality of \( \eta_L > 0 \) for the case of \( \beta_g < \beta_n \) and \( \hat{\Delta}_A \in (0, 1) \) can be shown in the same way.

Thus, with \( \hat{\Delta}_B \geq 0 \), the designer leaves a positive prize to a loser if and only if

\[
\begin{align*}
\alpha_g > \alpha_g^* \\
\alpha_n \in (0, \beta_n \hat{\Delta}_B)
\end{align*}
\]

When \( \alpha_n > \hat{\Delta}_B \beta_n \), the designer always prefers the \((1, \hat{\Delta}_B)\) allocations to the \((\hat{\Delta}_A, \hat{\Delta}_B)\) schedule (Lemma 8). Then, one must compare \( J(1, \hat{\Delta}_B) \) and \( J(1, 0) \) directly:

\[
J(1, \hat{\Delta}_B) > J(1, 0) \iff \alpha_g > \hat{r}_1 > 0 \tag{16}
\]

Condition (16) was derived and analyzed above.

**Lemma 11.** For \( \beta_g > 3\beta_n \) and \( \hat{\Delta}_B \geq 0 \) there exists \( \alpha_n^* > \hat{\Delta}_B \beta_n > 0 \) such that \( \hat{r}_1 > \alpha_n + \hat{\Delta}_B (\beta_g - \beta_n) \) for any \( \alpha_n > \hat{\Delta}_A \) and \( \hat{r}_1 < \alpha_n + \hat{\Delta}_B (\beta_g - \beta_n) \), otherwise. For \( \beta_g \in (0, 3\beta_n) \) and \( \hat{\Delta}_B \geq 0 \) the inequality \( \hat{r}_1 < \alpha_n - \hat{\Delta}_B \min (\beta_n - \beta_g, \beta_g - \beta_n) \) holds for any \( \alpha_n \geq 0 \).

**Proof.** Define \( f_1 (\alpha_n) = \hat{r}_1 - (\alpha_n + \hat{\Delta}_B (\beta_g - \beta_n)) \) and \( f_2 (\alpha_n) = \hat{r}_1 - (\alpha_n + \hat{\Delta}_B (\beta_n - \beta_g)) \).

First, take \( \beta_g \geq \beta_n \) and solve \( f_1 (\alpha_n) = 0 \) with respect to \( \alpha_n \):

\[
 f_1 (\alpha_n) = 0 \iff \alpha_n = k_i, i = \{1, 2\}
\]

where \( k_{1,2} = \frac{\hat{\Delta}_B (\pm \sqrt{\beta_n (\beta_n + \beta_g) (8 \beta_g^2 - 15 \beta_n \beta_g + 9 \beta_n) - 3 \beta_n^2 - 5 \beta_n})}{2 (\beta_g - 3 \beta_n)} \), \( k_1 < k_2 \). Different outcomes emerge.
• $\beta_g \in [\beta_n, 3\beta_n) \Rightarrow k_2 < 0$ and $f_1(\alpha_n) < 0$ for any $\alpha_n \geq 0$

• $\beta_g > 3\beta_n \Rightarrow k_1 < 0, k_2 > \hat{\Delta}_B \beta_n > 0$ and $f_1(\alpha_n) \leq 0$ for $\alpha_n \in [0, k_2)$, $f_1(\alpha_n) > 0$ for $\alpha_n > k_2$

Taking $\alpha_n^* = k_2$ for $\beta_g > 3\beta_n$, we get the first statement of the lemma.

Second, consider the case of $\beta_g < \beta_n$ and solve $f_2(\alpha_n) = 0$ with respect to $\alpha_n$:

$$f_2(\alpha_n) = 0 \Leftrightarrow \alpha_n = \frac{2\hat{\Delta}_B \beta_n (\beta_n - \beta_g)}{\beta_g - 3\beta_n} < 0 \text{ or } \alpha_n = -\hat{\Delta}_B \beta_n$$

Since both roots are negative, $f_2(\alpha_n)$ does not change its sign in $\alpha_n \geq 0$, and $f_2(\alpha_n) < 0$ holds for any $\alpha_n \geq 0$.

With Lemma 11, we can specify when the designer prefers to leave good $B$ out under $\hat{\Delta}_B \geq 0$:

$$J(1, 0) > J(1, \hat{\Delta}_B) > J(\hat{\Delta}_A, \hat{\Delta}_B) \Leftrightarrow \begin{cases} \alpha_g \in \left( \alpha_n - \hat{\Delta}_B (\beta_n - \beta_g), \hat{\Delta}_B \right) \\ \alpha_n > \alpha_n^*, \beta_g > 3\beta_n \end{cases}$$

$$J(1, \hat{\Delta}_B) > J(1, 0) > J(\hat{\Delta}_A, \hat{\Delta}_B) \Leftrightarrow \begin{cases} \alpha_g > \hat{\Delta}_B \beta_n \\ \alpha_n > \alpha_n^*, \beta_g > 3\beta_n \end{cases}$$

Next, assume $\hat{\Delta}_B < 0$. Any allocation with $\Delta_A < 0$ leads to negative winning benefits and destroys incentives to compete. The conditions, under which type-feasibility, $\hat{\Delta}_A \in [0, 1]$, and $\frac{\partial J^+(\hat{\Delta}_A, \hat{\Delta}_B)}{\partial \Delta_A} < 0$ hold simultaneously, are

$$\begin{cases} \alpha_g \in \left( \max \{ \alpha_n + (\beta_n - \beta_g) \hat{\Delta}_B, 3\alpha_n \}, \alpha_n \frac{\beta_n}{\beta_g} \right) \\ \alpha_n > -\beta_n \hat{\Delta}_B \\ \beta_g > 3\beta_n \end{cases}$$

**Lemma 12.** For $\beta_g > 7\beta_n$ and $\hat{\Delta}_B < 0$ there exist valuation profiles such that the designer prefers to leave a positive prize for a loser.

**Proof.** First, we compare $J(\hat{\Delta}_A, \hat{\Delta}_B)$ and $J(1, \hat{\Delta}_B)$. The case of $J(\hat{\Delta}_A, \hat{\Delta}_B) > J(1, \hat{\Delta}_B)$ is characterized by condition (13) and roots $s_1$ and $s_2$ (Lemma 8) where

$$s_1 < s_2 \Leftrightarrow \alpha_n \in \left( \frac{\beta_n^2}{2}, \hat{\Delta}_B \frac{\beta_n - \beta_g}{2} \right)$$
For $\beta_g > 3\beta_n$, this set has a non-empty intersection with $\alpha_n > -\beta_n \hat{\Delta}_B$. Also, $\alpha_n \in \left[0, \hat{\Delta}_B (\frac{\beta_n - \beta_g}{2})\right]$ implies $\{\alpha_n + (\beta_n - \beta_g) \hat{\Delta}_B > 3\alpha_n\}$. Combining all conditions together, $J \left(\tilde{\Delta}_A, \hat{\Delta}_B\right) > J \left(1, \hat{\Delta}_B\right)$ holds if and only if
\[
\begin{cases}
\alpha_g \in \left(\alpha_n + (\beta_n - \beta_g) \hat{\Delta}_B, \min\left\{\alpha_n \frac{\beta_n}{\beta_n}, s_2\right\}\right) \\
\alpha_n \in \left(-\hat{\Delta}_B \beta_n, \hat{\Delta}_B \left(\frac{\beta_n - \beta_g}{2}\right)\right) \\
\beta_g > 3\beta_n
\end{cases}
\]

(17)

Second, we investigate when $J \left(\tilde{\Delta}_A, \hat{\Delta}_B\right) > J \left(1, 0\right)_{\{\alpha_g > \alpha_n\}}$ holds:
\[J \left(\tilde{\Delta}_A, \hat{\Delta}_B\right) > J \left(1, 0\right)_{\{\alpha_g > \alpha_n\}} for \hat{\Delta}_B < 0 \iff \alpha_g \in [0, \bar{r}_1]\]
where $\bar{r}_1$ was defined in Lemma 9. This condition delivers a non-empty intersection with (17) if and only if
\[
\begin{cases}
\alpha_n \in \left(-\hat{\Delta}_B \beta_n, \min\left\{\frac{2\hat{\Delta}_B \beta_n (\beta_n - \beta_g)}{\beta_g - 3\beta_n}, \hat{\Delta}_B (\beta_n - \beta_g)\right\}\right) \\
\beta_g > 7\beta_n
\end{cases}
\]

Lemma 11 illustrates that the designer indeed assigns $A^L > 0$ in the bundle.

Using the proof of Lemma 12, one can also show when the designer prefers to leave good $B$ out and run a one-dimensional contest over item $A$. 

\[\square\]
Theorem 1. For any valuation profile \( r \in R \) the designer uses goods’ endowments completely and either

- Leaves a positive losing prize at least in one dimension or
- Gives both items to a winner.

Proof. Let \( r = \{\alpha_g, \alpha_n, \beta_g, \beta_n\} \). A complete characterization of the designer’s problem looks as follows:

\[
L \left( A^W, A^L, B^W, B^L, \{\eta_i\}_{i=W}^L, \{\kappa_i\}_{i=W}^L, \{\lambda_i\}_{i=B}^\theta \right) = J \left( A^W, A^L, B^W, B^L \right) + \eta_W A^W + \eta_L A^L + \kappa_W B^W + \kappa_L B^L + \lambda_A \left( 1 - A^W - A^L \right) + \lambda_B \left( 1 - A^W - A^L \right)
\]

The designer’s objective function can be rewritten in terms of prize spreads:

\[
J \left( A^W, A^L, B^W, B^L \right) \equiv J \left( \Delta_A, \Delta_B \right)
\]

We call player \( i, i = \{1, 2\} \) type \( g \) if and only if \( t_i(1, 1) > t_{-i}(1, 1) \).

Lemma 1. The designer’s problem has no interior solution.

Proof. To prove the statement, compute the hessian of \( J \left( \Delta_A, \Delta_B \right) \):

\[
H = \begin{bmatrix}
\frac{\Delta_A^2 (\beta_g \alpha_n - \beta_n \alpha_n)^2}{t_g^2} & \frac{\Delta_A \Delta_B (\beta_n \alpha_g - \beta_g \alpha_n)^2}{t_g^3} \\
-\frac{\Delta_A \Delta_B (\beta_n \alpha_g - \beta_g \alpha_n)^2}{(\alpha_g \alpha_A + \beta_n \Delta_B)^3} & \frac{\Delta_B^2 (\beta_n \alpha_g - \beta_g \alpha_n)^2}{t_g^3}
\end{bmatrix}
\]

Since \( H \) is not negative definite, any prize allocation that satisfies at least one first order condition corresponds to an interior minimum.

\[\square\]

Corollary 1. \( \frac{\partial J}{\partial \Delta_A} (\cdot) \) and \( \frac{\partial J}{\partial \Delta_B} (\cdot) \) increase in \( \Delta_A \) and \( \Delta_B \), respectively.

Since the designer can change the prize allocation in both dimensions, one does not need to consider reward schedules \((\Delta_A, 0)\) and \((0, \Delta_B)\), where one good is left out, separately. Fixing the prize scheme in one of the dimensions, define \( \tilde{\Delta} \)'s in items \( A \) and \( B \) such that contestants’ types equalize:

\[
t_g (\cdot) = t_n (\cdot) \iff \tilde{\Delta}_A = \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n} \Delta_B \text{ or } \tilde{\Delta}_B = \frac{\alpha_n - \alpha_g}{\beta_g - \beta_n} \Delta_A
\]

and the types are well-defined if and only if

\[
t_j \left( \Delta_A, \tilde{\Delta}_B \right) \geq 0 \iff \Delta_A \geq -\frac{\beta_j}{\alpha_j} \Delta_B \text{ or } \Delta_B \geq -\frac{\alpha_j}{\beta_j} \Delta_A, \ j = \{g, n\}
\]

It is easy to prove that \( \tilde{\Delta}_A \) (\( \tilde{\Delta}_B \)) cannot be feasible for \( \alpha_g < \alpha_n \ (\beta_g < \beta_n) \):

\[
\tilde{\Delta}_A \leq 1 \iff \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n} \Delta_B \leq 1 \iff \alpha_g \leq \alpha_n + (\beta_n - \beta_g) \Delta_B
\]

\[
\tilde{\Delta}_B \leq 1 \iff \frac{\alpha_n - \alpha_g}{\beta_g - \beta_n} \Delta_A \leq 1 \iff \beta_g \leq \beta_n + (\alpha_n - \alpha_g) \Delta_A
\]
Both inequalities contradict the definition of type \( g \) provided above. Thus, a necessary condition for feasibility of \( \Delta_A = \bar{\Delta}_A \) (\( \Delta_B = \bar{\Delta}_B \)) is \( \alpha_g > \alpha_n \) (\( \beta_g > \beta_n \)).

**Definition.** \( \bar{\Delta}_k, k = \{ A, B \} \) is type-feasible if and only if \( \max \{ \Delta^g_k, \Delta^n_k \} < \bar{\Delta}_k \).

The proof of Proposition 2 (**Lemma 2**) states that \( \frac{\partial J}{\partial \Delta_A} (\cdot) \) has a jump point at \( \Delta_A = \bar{\Delta}_A \) for any \( \alpha_g \neq \alpha_n \). A similar result holds for \( \frac{\partial J}{\partial \Delta_B} (\cdot) \) and \( \beta_g \neq \beta_n \).

**Lemma 2.** \( \frac{\partial J}{\partial \Delta_B} (\cdot) \) has a jump point at \( \Delta_B = \bar{\Delta}_B \) for any \( \beta_g \neq \beta_n \).

**Proof.** The limits of \( \frac{\partial J}{\partial \Delta_B} (\cdot) \) from the left and from the right of \( \Delta_B = \bar{\Delta}_B \) are

\[
\lim_{\Delta_B \to -\bar{\Delta}_B} \frac{\partial J}{\partial \Delta_B} (\cdot) = \frac{3\beta_g - \beta_n}{2}
\]

\[
\lim_{\Delta_B \to +\bar{\Delta}_B} \frac{\partial J}{\partial \Delta_B} (\cdot) = \frac{3\beta_n - \beta_g}{2}
\]

These values coincide if and only if \( \beta_g = \beta_n \). Thus, \( \frac{\partial J}{\partial \Delta_B} (\cdot) \) has a jump point at \( \Delta_B = \bar{\Delta}_B \) for any \( \beta_g \neq \beta_n \).

We treat \( \frac{\partial J}{\partial \Delta_A} (\cdot) \) and \( \frac{\partial J}{\partial \Delta_B} (\cdot) \) as right-continuous function and denote their derivatives to the left and to the right of \( \bar{\Delta} \)’s with the “−” and “+” signs, respectively.

**Lemma 3.** For any valuation profile such that \( \bar{\Delta}_A \) (\( \bar{\Delta}_B \)) is not type-feasible \( \frac{\partial J}{\partial \Delta_A} \) (\( \max \{ \Delta^g_A, \Delta^n_A \}, \Delta_B \)) > 0 \( \frac{\partial J}{\partial \Delta_B} (\Delta_A, \max \{ \Delta^g_B, \Delta^n_B \}) > 0 \).

**Proof.** The proof of Proposition 2 (**Lemma 3**) shows that there is no type-feasibility in dimension \( A \) if and only if \( \max \{ \Delta^g_A, \Delta^n_A \} = \Delta^n_A \). One can apply similar arguments and prove that \( \max \{ \Delta^g_B, \Delta^n_B \} = \Delta^n_B \) in the absence of type-feasibility over good \( B \).

Next, evaluate \( \frac{\partial J}{\partial \Delta_A} (\Delta^A_A, \Delta_B) \) and \( \frac{\partial J}{\partial \Delta_B} (\Delta_A, \Delta^n_B) \):

\[
\frac{\partial J}{\partial \Delta_A} (\Delta^A_A, \Delta_B) = \frac{\alpha_n}{2} > 0
\]

\[
\frac{\partial J}{\partial \Delta_B} (\Delta_A, \Delta^n_B) = \frac{\beta_n}{2} > 0
\]

and the statement of the lemma follows.

**Lemma 4.** For any valuation profile such that \( \bar{\Delta}_k, k = \{ g, n \} \) is type-feasible the objective function \( J (\cdot) \) strictly increases in \( \Delta_k \) for \( \Delta_k \in \left[ \bar{\Delta}_k^g, \bar{\Delta}_k \right] \). \( J (\cdot) \) strictly decreases or is non-monotone in \( \Delta_k \) for \( \Delta_k \in \left[ \bar{\Delta}_k, 1 \right] \) if and only if \( \frac{\partial J}{\partial \Delta_k} \) \( \bar{\Delta}_k \) < 0; otherwise, \( J (\cdot) \) strictly increases in \( \Delta_k \) for \( \Delta_k \in \left[ \bar{\Delta}_k, 1 \right] \).

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Proof. The proof of Proposition 2 (Lemma 4) characterizes the case of type-feasibility in dimension $A$ ($k = A$). The same set of arguments can be used to prove the lemma for $k = B$ when $\beta_g > 3\beta_n$ is necessary to support non-monotonicity of $J(\cdot)$ in $\Delta_k$.

Enough heterogeneity in contestants’ valuations over a particular dimension ($\alpha_g > 3\alpha_n$ or / and $\beta_g > 3\beta_n$) is necessary (but not sufficient) to make a positive losing prize optimal. Given Lemma 1 and Lemma 4, the optimal reward schedule is a corner solution of the designer’s problem. Then, the following alternatives must be compared directly when type-feasibility holds:

$$(1, 1) \text{ vs. } \left(1, \max\left\{\min\{\tilde{\Delta}_A, 1\}, -1\right\}\right) \text{ vs. } \left(\max\left\{\min\{\tilde{\Delta}_A, 1\}, -1\right\}, 1\right) \text{ vs. } \left(-1, \max\left\{\min\{\tilde{\Delta}_B, 1\}, -1\right\}\right) \text{ ...}$$

A single allocation that cannot feature positive losing prizes is the $(1, 1)$ schedule. To prove the optimality of $\Delta_A \neq 1$ or / and $\Delta_B \neq 1$, it is sufficient to show when $(1, 1)$ is dominated by at least one reward schedule with $\Delta_A < 1$ or / and $\Delta_B < 1$. Take Proposition 2 and fix $\tilde{\Delta}_B = 1$. Then, there must exist valuation profiles such that the designer prefers to use both goods and leaves a positive losing prize in dimension $A$. With the details from the proof of Proposition 2, we construct an example. Assume $\beta_g > \beta_n$ and $\frac{\partial J}{\partial \tilde{\Delta}_B} + \left(\tilde{\Delta}_B\right) > 0 \iff \beta_g \in (\beta_n, 3\beta_n)$. Then, Corollary 1, Lemma 3 and Lemma 4 imply that $\frac{\partial J}{\partial \tilde{\Delta}_B} (\cdot) > 0$ holds for any feasible prize allocation, and $\Delta_B = 1$ is optimal. Next, take $\tilde{\Delta}_B = 1$ and consider the case when $\tilde{\Delta}_A = \frac{\beta_n - \beta_g}{\alpha_g - \alpha_n}$ is type-feasible.

The designer prefers the $\left(\tilde{\Delta}_A, 1\right)$ allocation to the $(1, 1)$ schedule if and only if

$$\left\{ \begin{array}{l} \alpha_g > \max\left\{\alpha_n + (\beta_g - \beta_n), 3\alpha_n, \alpha_n \frac{\beta_g - \beta_n}{\alpha_g - \alpha_n}, \alpha_g^*\right\} \\ \alpha_n \in (0, \beta_n), \beta_g \in (\beta_n, 3\beta_n) \end{array} \right.$$ 

where all conditions and $\alpha_g^*$ were defined in the proof of Proposition 2. One can construct examples for other valuation profiles similarly. To show that the designer uses goods’ endowments completely, the proof of Proposition 2 (Lemma 10) applies.

\[\square\]
Theorem 2. There exists a non-empty subset of $R, R^h$, such that for any $\{\alpha_g, \alpha_n, \beta_g, \beta_n\} \in R^h$ the designer prefers one contest with bundled prizes to two separate contests with unbundled prizes.

Proof. To show the existence of $R^h \neq \emptyset$ we refer to the proofs of Proposition 2 and Theorem 1. First, consider the case when the designer prefers $\Delta_A = \max \{\tilde{\Delta}_A, -1\}$ and gives item $B$ to a winner ($\Delta_B = 1$, and $\beta_g \in (\beta_n, 3\beta_n)$ is assumed):

1. $\tilde{\Delta}_A \leq -1 \iff \alpha_g < \alpha_n - \beta_n + \beta_g$, i.e. $\tilde{\Delta}_A$ is not feasible.

The optimality of $(1, 1)$ requires

\[
\begin{cases}
\alpha_g \in \left(\max \{r_1, 3\alpha_n\} \cap \alpha_n - (\beta_n - \beta_g) \tilde{\Delta}_B\right) \\
\alpha_n \in [0, q_1], \beta_g \in (\beta_n, 3\beta_n)
\end{cases}
\]

where all roots and conditions were defined in the proof of Proposition 2 ($\tilde{\Delta}_B = 1$ is taken). The $(1, 1)$ schedule dominates two separate contests over dimensions $A$ and $B$ if and only if

\[
J(-1, 1) > J(1, 0)_{\{\alpha_g > \alpha_n\}} + J(0, 1)_{\{\beta_g > \beta_n\}} \iff
\]

\[
(\beta_n^2 + 2\beta_g\alpha_n)\alpha_g^2 - 2\beta_g\alpha_n (\beta_g + \beta_n - \alpha_n) \alpha_g - \beta_g^2 \alpha_n^2 > 0
\]

The underlying square equation always has two real roots, $h_1$ and $h_2$, $h_1 < 0 < h_2$. Then, $J(-1, 1) > J(1, 0)_{\{\alpha_g > \alpha_n\}} + J(0, 1)_{\{\beta_g > \beta_n\}}$ holds if and only if $\alpha_g > h_2$, and this set has a non-empty intersection with (18) if and only if $h_2 < \alpha_n - \beta_n + \beta_g$.

Lemma 1. For $\tilde{\Delta}_A \leq -1$ there exists $\alpha_n^h \in (0, \beta_n)$ such that $h_2 < \alpha_n - \beta_n + \beta_g$ for any $\alpha_n \in (0, \alpha_n^h)$.

Proof. Define $f^h(\alpha_n) = h_2 - \alpha_n - (\beta_g - \beta_n)$. Solving $f^h(\alpha_n) = 0$ with respect to $\alpha_n$ delivers:

\[
f^\phi(\alpha_n) = 0 \iff \alpha_n = \alpha_n^i, \ i = \{1, ..., 4\}
\]

\[
\alpha_n^1 = \beta_n, \ \alpha_n^2 = -\frac{\beta_g^2}{2\beta_g}
\]

\[
\alpha_n^j = \frac{-3\beta_g^2 - \beta_n^2 + 4\beta_g\beta_n + (\beta_g - \beta_n) (\beta_g + \beta_n)(9\beta_g + \beta_n)}{8\beta_g}, \ j = \{3, 4\}
\]

\[
\alpha_n^3 < 0 < \alpha_n^4 < \beta_n
\]

where $f^h(\alpha_n) < 0$ for $\alpha_n \in (0, \alpha_n^4)$ and $f^h(\alpha_n) > 0$ for $\alpha_n \in (\alpha_n^4, \beta_n)$. Taking $\alpha_n^h = \alpha_n^4$ gives the statement of the lemma.
Thus, the designer prefers the \((-1, 1)\) allocation to two separate contests if and only if

\[
r \in R^b_1 = \begin{cases} 
\alpha_g \in (\max \{r_1, 3\alpha_n, b_2\}, \alpha_n - \beta_n + \beta_g) \\
\alpha_n \in (0, \min \{q_1, \alpha_n^h\}), \beta_g \in (\beta_n, 3\beta_n)
\end{cases}
\]

2. \(\tilde{\Delta}_A \in (-1, 1) \iff \alpha_g > \alpha_n + \beta_g - \beta_n\) for \(\beta_g \in (\beta_n, 3\beta_n)\), i.e. \(\tilde{\Delta}_A\) is feasible.

With two-dimensional rewards, the designer assigns a positive losing prize in dimension \(A\) if and only if

\[
\alpha_g > \max \{\alpha_n + \beta_g - \beta_n, 3\alpha_n, \alpha_n \frac{\beta_g}{\beta_n}, \alpha_g^*\}
\]
\[
\alpha_n \in (0, \beta_n), \beta_g \in (\beta_n, 3\beta_n)
\]

(19)

The \((\tilde{\Delta}_A, 1)\) bundle dominates two separate contests with single-item prizes if and only if:

\[
J(\tilde{\Delta}_A, 1) > J(1, 0)_{\{\alpha_g > \alpha_n\}} + J(0, 1)_{\{\beta_g > \beta_n\}} \iff \gamma_1 \alpha_g^2 + \gamma_2 \alpha_g + \gamma_3 > 0
\]
\[
\gamma_1 = -\beta_n^2 + \beta_g \beta_n - \beta_g \alpha_n, \gamma_2 = \alpha_n (-2\beta_g^2 + \beta_g \beta_n + \beta_n^2)
\]
\[
\gamma_3 = \beta_g^2 \alpha_n^3
\]
\[
\gamma_1 > 0 \iff \alpha_n < \frac{\beta_n(\beta_g - \beta_n)}{\beta_g}, \gamma_2 < 0 \forall \beta_g > \beta_n
\]

The corresponding square equation always has two real roots, \(r_1^{\gamma_1}\) and \(r_2^{\gamma_1}\), \(r_1^{\gamma_1} < r_2^{\gamma_1}\). Take the case of \(\alpha_n \in \left(0, \frac{\beta_n(\beta_g - \beta_n)}{\beta_g}\right)\) when \(\gamma_1 > 0\). This implies \(r_1^{\gamma_1} > 0\), and the bundle is optimal if and only if \(\alpha_g \in (0, r_1^{\gamma_1}) \cup (r_2^{\gamma_1}, \infty)\). Then, a sufficient condition to guarantee the optimality of the \((\tilde{\Delta}_A, 1)\) allocation over two separate contests is

\[
r \in R^b_2 = \begin{cases} 
\alpha_g > \max \{\alpha_n + \beta_g - \beta_n, 3\alpha_n, \alpha_n \frac{\beta_g}{\beta_n}, \alpha_g^*, r_1^{\gamma_1}\} \\
\alpha_n \in \left(0, \frac{\beta_n(\beta_g - \beta_n)}{\beta_g}\right), \beta_g \in (\beta_n, 3\beta_n)
\end{cases}
\]

Next, we show when the \((1, 1)\) bundle dominates two simultaneous competitions over dimensions \(A\) and \(B\). In the beginning, assume \(\alpha_g > \alpha_n\) and \(\beta_g > \beta_n\). One contest with bundled prizes is preferred to two separate competitions if and only if

\[
J(1, 1) > J(1, 0)_{\{\alpha_g > \alpha_n\}} + J(0, 1)_{\{\beta_g > \beta_n\}} \iff
\]

\[
\iff -\frac{(\beta_g \alpha_n - \beta_n \alpha_g)^2}{2 \beta_g \alpha_g \alpha_n (\alpha_n + \beta_g)} > 0
\]

This condition never holds for \(\beta_g, \alpha_g > 0\). Thus, if type \(g\) values both goods more than his opponent, the “Winner-Takes-All” bundle never dominates two simultaneous competitions.
Now, we analyze the case of $\alpha_g > \alpha_n$ and $\beta_g < \beta_n$. To support the optimality of $\Delta_A = 1$, we impose $\alpha_g \in (\alpha_n, 3\alpha_n)$ (see the proof of Theorem 1). The “Winner-Takes-All” bundle dominates other feasible two-dimensional allocations if and only if

$$\begin{align*}
\alpha_g &\in (\alpha_n + \beta_n - \beta_g, 3\alpha_n), \quad \alpha_n > \frac{\beta_n - \beta_g}{2} \\
\beta_g &\in [0, \beta_n)
\end{align*}$$

(20)

The designer prefers the (1, 1) allocation to two simultaneous contests with single-item rewards if and only if

$$J(1, 1) > J(0, 1)_{\{\alpha_g > \alpha_n\}} + J(0, 1)_{\{\beta_g < \beta_n\}} \iff \eta_1 \alpha_g^2 + \eta_2 \alpha_g + \eta_3 > 0$$

$$\eta_1 = -\beta_g^2 - \beta_g \beta_n + \beta_n^2, \quad \eta_2 = (\beta_n - \beta_g)(\beta_g + \beta_n)^2 + 2\alpha_n\beta_n^2 > 0$$

$$\eta_3 = -\alpha_n^2 \beta_n \beta_g < 0$$

$$\eta_1 > 0 \iff \beta_g \in [0, \bar{\beta}_g), \quad \bar{\beta}_g = \frac{\sqrt{7}-1}{2}\beta_n < \beta_n$$

For $\beta_g < \beta_n$, the underlying square equation always has two real roots, $r_1^\eta$ and $r_2^\eta$. Take the case of $\beta_g \in \left[\frac{\beta_n}{2}, \bar{\beta}_g\right)$ when $\eta_1 > 0$. Then, $r_1^\eta < 0 < r_2^\eta$ holds, and the (1, 1) bundle dominates two single-item contests if and only if $\alpha_g > r_2^\eta$. The condition $r_2^\eta < \alpha_n + \beta_n - \beta_g$ holds for any $\alpha_n > 0$ and $\beta_g < \beta_n$. Then, $\alpha_g \in (\alpha_n + \beta_n - \beta_g, 3\alpha_n)$ implies $\alpha_g > r_2^\eta$, and the designer prefers the “Winner-Takes-All” bundle to other alternative reward schedules if and only if

$$r \in R_3^b = \left\{ \begin{array}{l}
\alpha_g \in (\alpha_n + \beta_n - \beta_g, 3\alpha_n), \quad \alpha_n > \frac{\beta_n - \beta_g}{2} \\
\beta_g \in [0, \beta_n)
\end{array} \right.$$

Taking a union of $R_1^b$, $R_2^b$, and $R_3^b$, we define a non-empty subset of $R^b$:

$$R_1^b \cup R_2^b \cup R_3^b \subseteq R^b$$

$\blacksquare$
Proposition 4. For \( \min \{t_1^g, t_2^g\} > 0 \) there exists a unique non-trivial equilibrium in monotonically increasing strategies such that:

- At least one type places an atom at zero: \( \exists i = \{1, 2\}, j = \{g, n\} : G_i(t_i^j, 0) > 0; \)
- There is no \( e > 0 \) played with a positive probability;
- Supports of \( G_i(t_i^g, e), i = \{1, 2\} \) have the same supremum: \( \sup(s_i^g) = \sup(s_2^g); \)
- Supports of \( G_i(t_i^g, e), i = \{1, 2\} \) have the same infimum, and it is equal to zero: \( \inf(s_i^g) = \inf(s_2^g) = 0. \)

Proof. To show the existence, I construct an equilibrium. Given properties derived in Siegel (2014), the equilibrium of this game must look like a partition where various types randomize on specific intervals. Another result states that this partition can be obtained using a “top-down” approach.

Let \( U_{i,l} \) correspond to the losing benefit of contestant \( i \) with type \( l, l = \{g, n\} \). Assume \( t_i^g > t_j^g, i, j = \{1, 2\} \). First, I show that \( t_j^g \leq 0 \) never supports the non-trivial equilibrium.

Lemma 1. For \( t_j^g \leq 0 \) there does not exist a non-trivial equilibrium.

Proof. Assume \( t_j^g \leq 0 \) chooses \( e_j^g > 0 \) in equilibrium. If he wins, the payoff becomes:

\[
\pi_{j,g}^w (e_j^g) = t_j^g + U_{j,g}^L - e_j^g
\]

Bidding \( e_j^g = 0 \), the contestant gets \( \pi_{j,g}^w(0) = t_j^g + U_{j,g}^L (\pi_{j,g}^L(0) = U_{j,g}^L) \) if wins (loses). Then \( \pi_{j,g}^L(0) \geq \pi_{j,g}^w(0) > \pi_{j,g}^w(e_j^g) \) for any \( e_j^g > 0 \), \( t_j^g \leq 0 \), and \( e_j^g > 0 \) cannot be in equilibrium. As a result, \( t_j^g \leq 0 \) must place \( p_i^0(t_j^g) = 1 \) at \( e = 0. \)

Next, consider \( t_j^g > 0 \) and characterize the top-interval of the partition where \( t_i^g \) and \( t_j^g \) play against each other. Take two effort choices, \( 0 < e_i^g < e_i^g \), corresponding to best responses of \( t_i^g \) and generating the same equilibrium payoff. Then it must be:

\[
P \left( t_i^g \right) \left( t_i^g + U_{i,g}^L \right) + P \left( t_j^g \right) \left( t_i^g G_j \left( t_j^g, e_i^g \right) + U_{i,g}^L \right) - e_i^g =
\]

\[
= P \left( t_j^g \right) \left( t_i^g + U_{i,g}^L \right) + P \left( t_j^g \right) \left( t_i^g G_j \left( t_j^g, e_i^g \right) + U_{i,g}^L \right) - e_i^g
\]

The first element of the sum reflects the case when \( t_i^g \) plays against \( t_j^g \) and wins with certainty because \( t_j^g \) never bids in the top interval. Matched against \( t_j^g, t_i^g \) succeeds with probability \( G_j \left( t_j^g, e \right) \)

(95) Simplification delivers

\[
\frac{G_j \left( t_j^g, e_i^g \right) - G_j \left( t_j^g, e_i^g \right)}{e_i^g - e_i^g} = \frac{1}{P \left( t_j^g \right) t_i^g}
\]
Thus, contradiction can obtain where the right-hand side is constant. Taking $\tilde{c}_i^g - e_i^g \to 0$ brings $g_i(t_i^g, e) = \frac{1}{\tilde{P}(t_i^g)t_i^g}$. Similarly, one can obtain $g_j(t_j^g, e) = \frac{1}{\tilde{P}(t_j^g)t_j^g}$.

Type $t_i^g$ ($t_j^g$) exhausts his bidding probability in the top-interval if its length ($L_{top}$) is equal to $P(t_i^g) \tilde{t}_j^g$, and $P(t_j^g) \tilde{t}_i^g)$. Since $t_j^g < t_i^g$ by assumption, it must be $L_{top} = P(t_i^g) \tilde{t}_j^g$, otherwise, there is a contradiction:

$$L_{top} = P(t_i^g) \tilde{t}_j^g \Rightarrow G_i(t_i^g, P(t_j^g) \tilde{t}_i^g) = \frac{t_i^g}{\tilde{t}_i^g} > 1$$

Thus, $t_i^g$ exhausts his bidding probability, and $t_j^g$ has $\left\{1 - \frac{t_j^g}{\tilde{t}_j^g}\right\}$ to expend in the game against $t_i^g$.

Moving downward, take the interval where $t_j^g$ exhausts his bidding probability, and $t_i^n$ has $\left\{1 - \frac{t_i^n}{\tilde{t}_i^n}\right\}$ to expend in the game against $t_j^n$. Several possibilities emerge:

1. $t_i^n > 0$.

Following similar steps, one can show:

$$g_i(t_i^n, e) = \frac{1}{\tilde{P}(t_i^n)t_i^n}, g_j(t_j^n, e) = \frac{1}{\tilde{P}(t_j^n)t_j^n}$$

Types’ bidding probabilities exhaust in the intervals of length $L_i^m = P(t_i^n) \tilde{t}_j^n$ for $t_i^n$ and $L_j^m = P(t_j^n) \tilde{t}_i^n$ for $t_j^n$. $t_j^n$ cannot expend more than $\left\{1 - \frac{t_j^n}{\tilde{t}_j^n}\right\}$. Then $L_i^m > L_j^m$ holds iff:

$$L_i^m > L_j^m \Leftrightarrow P(t_j^n) < \frac{t_i^n \tilde{t}_j^n}{t_i^n (t_i^n - t_j^n) + t_j^n \tilde{t}_i^n} \equiv \tilde{T}, \tilde{T} \in [0, 1)$$

(a) $P(t_j^n) \in \left[0, \tilde{T}\right) \Rightarrow L_m = L_j^m$. This implies that type $t_j^n$ exhausts his bidding probability in $L_m$ and $t_i^n$ expends $\left\{\frac{P(t_j^n)(t_i^n - t_j^n)}{t_j^n t_i^n \tilde{P}(t_i^n)}\right\}$. As a result, $t_i^n$ can compete against $t_j^n$ in the lowest segment of the equilibrium partition that has length $L_b$.

In the beginning, take the case of $t_j^g > 0$. Then the strategies of $n$ types become:

$$G_i(t_i^n, e) = \frac{e}{\tilde{P}(t_i^n)t_i^n}, G_j(t_j^n, e) = \frac{e}{\tilde{P}(t_j^n)t_j^n}$$

$t_i^n$ and $t_j^n$ exhaust their bidding probabilities in intervals of length $L_i^b = P(t_i^n) \tilde{t}_j^n$ and $L_j^b = t_j^n \left(\frac{P(t_j^n)(t_i^n - t_j^n)}{t_j^n t_i^n \tilde{P}(t_i^n)}\right)$, respectively:

$$L_i^b < L_j^b \Leftrightarrow P(t_j^n) > \frac{t_i^n \tilde{t}_j^n (t_i^n - t_j^n)}{t_i^n \tilde{t}_j^n (t_i^n - t_j^n) + t_i^n \tilde{t}_i^n (t_i^n - t_j^n)} \equiv \hat{T}, \hat{T} \in (0, 1), \hat{T} < \tilde{T}$$

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88I use $P(t_i^n) = P(t_j^n) = 1 - P(t_j^n) \tilde{t}_j^n$ to simplify the expression.
Consider these options separately:

- $L_i^b < L_b^1 \Rightarrow L_b = L_i^b$, and $t^n_i$ exhausts his bidding probability on the interval. **Lemma 2.** inf $(s^n_1) = \inf (s^n_2) = 0$ in equilibrium.

**Proof.** The argument is similar to one provided in the proof of Proposition 1, **Lemma 2**.

As $t^n_j$ still has some bidding probability left, he places $p^0_i (t^n_j)$ at $e = 0$, $p^0_i (t^n_j) = 1 - \frac{\epsilon^j_i}{P(t^n_j)^n} t^n_j \Bigg( \frac{P(t^n_i)e^n_i - P(t^n_j)(t^n_i - t^n_j)}{e^n_i} \Bigg)$. □

- $L_i^b > L_b^1 \Rightarrow L_b = L_i^b$, and $t^n_j$ exhausts his bidding probability in the interval. **Lemma 2** holds, and $t^n_i$ must place $p^0_i (t^n_i) = \bigg(1 - \frac{P(t^n_i)(t^n_i - t^n_j)}{e^n_i t^n_j} P(t^n_j)^n\bigg) e^n_i$ at $e = 0$.

To sum up, for $P (t^n_j) \in [0, \bar{t})$, $t^n_j > 0$ the equilibrium partition $T_i, i \in \{1, 2, 3\}$ and contestants’ strategies can be characterized as follows:

- $T_1 = L_b$: $- P (t^n_j) \in \left(\max \{0, \bar{t}\}, \bar{T}\right)$: with probability $\left\{1 - \frac{P(t^n_i)(t^n_i - t^n_j)}{e^n_i t^n_j} P(t^n_j)^n\right\}$ $t^n_i$ randomizes uniformly on $[0, T_1]$; $t^n_j$ uses a strategy including uniform randomization on $(0, T_1]$ and the atom $p^0_i (t^n_j) \in (0, 1)$ placed at $e = 0$;
- $P (t^n_j) \in \left(0, \max \{0, \bar{t}\}\right)$: $t^n_j$ randomizes uniformly on $[0, T_1]$ with probability $1$; $t^n_i$ uses a strategy including uniform randomization on $(0, T_1]$ and the atom $p^0_i (t^n_j) \in (0, 1)$ placed at $e = 0$;

- $T_2 = T_1 + L_m$: $t^n_i$ and $t^n_j$ randomize uniformly on $(T_1, T_2]$ with probabilities $\left\{\frac{P(t^n_j)(t^n_i - t^n_j)}{e^n_i t^n_j} P(t^n_j)^n\right\}$ and $\left\{1 - \frac{\epsilon^j_i}{\epsilon^j_i t^n_j}\right\}$, respectively;

- $T_3 = T_2 + L_{top}$: $t^n_i$ and $t^n_j$ randomize uniformly on $(T_2, T_3]$ with probabilities $1$ and $\left\{\frac{\epsilon^j_i}{\epsilon^j_i t^n_j}\right\}$, respectively.

Further suppose $t^n_j \leq 0$. In this case type $t^n_j$ places the atom $p^0_j (t^n_j) = 1$ at $e = 0$ (the argument is similar to **Lemma 1**). **Lemma 3** characterizes the strategy of $t^n_i$:

**Lemma 3.** For $P (t^n_j) \in [0, \bar{t})$, $t^n_j \leq 0$ type $n$ of contestant $i$ ($t^n_i$) places $p^0_i (t^n_i) = 1 - \frac{P(t^n_i)(t^n_i - t^n_j)}{e^n_i t^n_j} P(t^n_j)^n \in (0, 1)$ at $e = 0$.

**Proof.** When $t^n_j \leq 0$, type $t^n_i$ chooses $e = 0$ with probability $1$ (the argument is similar to **Lemma 1**). Suppose there exists $y > 0$ such that $t^n_j$ never bids in $[0, y]$, but $t^n_i$ randomizes uniformly on $[0, y]$ when plays against $t^n_j$ in equilibrium. For any $e^n_i \in (0, y]$ type $t^n_i$ wins with certainty and gets $\pi^W_{i,n} (e^n_i) = t^n_i + U^L_{i,n} - e^n_i$. Take $e^n_i, \bar{e}^n_i \in (0, y]$,
$e_i^n > \tilde{e}_i^n$. Then $\pi_{i,n}^W(e_i^n) > \pi_{i,n}^W(\tilde{e}_i^n)$, and $\tilde{e}_i^n$ dominates $e_i^n$. That cannot be in equilibrium.

As a result, type $t_i^n$ must place the atom at $e \geq 0$ when plays against $t_j^n \leq 0$.

Further assume $t_i^n$ places the atom at $e = q > 0$ in equilibrium and wins against $t_j^n$ with certainty. Take $\varepsilon > 0$ small enough, $\varepsilon \in (0, q)$. Then $\pi_{i,n}^W(q - \varepsilon) > \pi_{i,n}^W(q)$ holds for any $\varepsilon > 0$, and there exists a profitable deviation. As a result, for $t_j^n \leq 0$ type $t_i^n$ never places the atom at $e > 0$ in equilibrium. Since there is no other type of contestant $j$ to compete with, $t_i^n$ must expend his bidding probability left by allocating

$$p_2^n(t_i^n) = 1 - \frac{p_1^n(t_i^n)(q - t_j^n)}{t_i^n P(t_j^n)} \in (0, 1) \text{ to } e = 0.$$

Summing up, for $P(t_j^n) \in [0, \bar{T})$, $t_j^n \leq 0$ the equilibrium partition $M_i$, $i = \{1, 2\}$ and contestants’ strategies look as follows:

- $M_1 = L_m$: $t_j^n$ and $t_i^n$ place $p_2(t_i^n) = 1$ and $p_2(t_j^n) \in (0, 1)$ at $e = 0$, respectively; $t_i^n$ and $t_j^n$ randomize uniformly on $(0, M_1]$ with corresponding probabilities \{1 - $p_2(t_i^n)$\} and \{1 - $\frac{t_j^n}{t_i^n}$\};

- $M_2 = M_1 + L_{top}$: $t_i^n$ and $t_j^n$ randomize uniformly on $(M_1, M_2]$ with probabilities 1 and \{1 - $\frac{t_j^n}{t_i^n}$\}, respectively.

(b) $P(t_j^n) \geq \bar{T} \Rightarrow L_m = L_m^i$. In this case type $t_j^n$ still has a positive bidding probability left, but $t_i^n$ exhausts his strategy in the lowest interval of the equilibrium partition. Since there is no other type of contestant $i$ to compete with, $t_j^n$ must place $p_2^n(t_j^n) = 1 - \frac{e_j^n(t_j^n P(t_j^n) + \tilde{e}_j^n P(t_j^n))}{P(t_j^n) t_i^n} \in (0, 1)$ at $e = 0$. Then the equilibrium partition is characterized by thresholds $D_1 = P(t_i^n) t_j^n$ and $D_2 = P(t_j^n) t_i^n + D_1 = \tilde{t}_j^n$.

**Lemma 4.** For $P(t_j^n) \geq \bar{T}$ type $n$ of contestant $j$ places $p_2^n(t_j^n) = 1$ at $e = 0$.

**Proof.** Suppose $t_j^n$ randomizes in $(0, h) \subseteq (0, D_1)$ in equilibrium. Then $t_j^n$ always loses against $t_i^n$ ($t_j^n$ bids on $[D_1, D_2]$), but can win with a positive probability if he faces $t_i^n$. Take $e_j^n, \tilde{e}_j^n \in (0, h)$; in equilibrium $t_j^n$ must be indifferent between all the points of this interval:

$$P(t_j^n) U_{j,n} + P(t_i^n) \left(t_i^n G_i(t_i^n, e_j^n) + U_{j,n}\right) - e_j^n =$$

$$= P(t_j^n) U_{j,n} + P(t_i^n) \left(t_i^n G_i(t_i^n, \tilde{e}_j^n) + U_{j,n}\right) - \tilde{e}_j^n.$$

When substitute for $G_i(t_i^n, e)$, the simplified expression becomes:

$$\left(e_j^n - \tilde{e}_j^n\right) \left[\frac{t_j^n}{t_i^n} - 1\right] = 0$$

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Since \( \left[ \frac{\alpha}{\eta_j} - 1 \right] < 0 \), this equality holds iff \( e_j^n = \tilde{e}_j^n \), a contradiction. Hence, \( t_j^n \) must place the atom of size 1 at \( e \geq 0 \).

Now assume \( t_j^n \) always plays \( e_j^n = q > 0 \), \( q \in (0, D_1) \) in equilibrium, and this results in expected payoff \( \pi_{j,n} (q) \):

\[
\pi_{j,n} (q) = P (t_i^n) U^L_{j,n} + P (t_i^n) (t_j^n G_i (t_i^n, q) + U^L_{j,n}) - q 
\]

Given that \( q \) is a best reply, it must be \( \pi_{j,n} (q) \geq \pi_{j,n} (0) = U^L_{j,n} \) where \( \pi_{j,n} (0) \) is a payoff when \( t_j^n \) chooses \( e_j^n = 0 \):

\[
\pi_{j,n} (q) \geq \pi_{j,n} (0) \iff q \left[ \frac{t_j^n}{t_q^n} - 1 \right] \geq 0 
\]

As \( \left[ \frac{t_j^n}{t_q^n} - 1 \right] < 0 \) always holds, the condition is never satisfied for \( q > 0 \): \( e_j^n = 0 \) dominates \( e_j^n = q \) for any \( q > 0 \). Hence, \( t_j^n \) must place \( p_0^0 (t_j^n) = 1 \) at \( e = 0 \) in equilibrium.

Overall, for \( P (t_j^n) \geq \tilde{T} \) the equilibrium partition and contestants’ strategies look become:

1. \((D_1, D_2)\): \( t_i^n \) and \( t_j^n \) randomize uniformly on the interval with probabilities 1 and \( \left\{ \frac{t_i^n}{t_j^n} \right\} \), respectively;
2. \((0, D_1)\): \( t_i^n \) and \( t_j^n \) randomize uniformly on the interval with probabilities 1 and \( \left\{ \frac{P(t_i^n) e_j^n}{P(t_i^n) e_j^n} \right\} \), respectively;
3. \( t_j^n \) and \( t_j^n \) place \( p_0^0 (t_j^n) \in (0, 1) \) and \( p_0^0 (t_j^n) = 1 \) at \( e = 0 \), respectively.

2. \( t_i^n \leq 0 \).

First, \( t_i^n \leq 0 \) will always place \( p_0^0 (t_i^n) = 1 \) at \( e = 0 \) in equilibrium (the argument is similar to Lemma 1). Then \( t_j^n \) competing against \( t_i^n \leq 0 \) must choose \( p_2^0 (t_j^n) = 1 - \frac{t_j^n}{t_i^n} \in (0, 1) \) to exhaust his bidding probability (the argument is similar to Lemma 3). Finally, \( t_j^n \) plays \( e = 0 \) with probability 1 (Lemma 4). As a result, the equilibrium partition consists of one interval \([0, L_{top}]\):

1. \( t_i^n \) and \( t_j^n \) randomize uniformly on \([0, L_{top}]\) with probabilities 1 and \( \left\{ \frac{t_i^n}{t_j^n} \right\} \), respectively;
2. \( t_j^n \) places \( p_2^0 (t_j^n) \in (0, 1) \) at \( e = 0 \); \( t_i^n \) and \( t_j^n \) always choose \( e = 0 \).

To show the uniqueness, I refer to Siegel (2014). He proved that the “top-down” algorithm delivers the unique equilibrium of the specified game if \( P (t_j | t_i) t_i \) increases in \( t_i \) for any \( t_j \), \( i, j = \{1, 2\} \):

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\[
P(t_j^g | t_i^g) t_i^g > P(t_j^g | t_i^n) t_i^n, \quad P(t_j^n | t_i^g) t_i^g > P(t_j^n | t_i^n) t_i^n, \quad i \neq j
\]

Since in this case realizations of \( \alpha_1 \) and \( \alpha_2 \) are independent, it must be \( P(t_j | t_i) = P(t_j) \). Then the condition reduces to \( t_i^g > t_i^n \) that holds by definition. Thus, the partitions characterized above correspond to a unique equilibrium of the game for given preferences, types’ probability distribution, and prize schedules.
Proposition 5. For $\beta_1 > \beta_2$ and $\alpha_1 > \alpha_2 + \beta_1 - \beta_2$ there exists a non-empty subset of $R_k^L$, $R_k^{L,s}$, such that for any probability-valuation profile in $R_k^{L,s}$ the designer uses both goods completely and assigns a positive losing prize in dimension $A$.

Proof. The problem of the designer was introduced in Proposition 2. Define $J (A^W, A^L, B^W, B^L) \equiv J (\Delta_A, \Delta_B)$. Depending on the probability-valuation profile and the prize scheme, the equilibrium partition in the game between contestants looks differently (see Proposition 4). Moreover, expected aggregate effort exhibits high-order non-linearity in prize spreads ($\Delta_A$ and $\Delta_B$), and $R_k^L$ cannot be characterized completely. To get analytical results, I look only at asymmetric preference profiles ($\bar{\alpha}_1 > \bar{\alpha}_2$, $\bar{\alpha}_1 < \beta_2$), assume $\alpha_1 > \alpha_2$ and characterize $\bar{R}_k^L$, a non-empty subset of $R_k^L$.

Suppose $\Delta_B = 1$ is optimal (verify this later). Also assume no switch in contestants’ identities for any feasible $\Delta_A$:

$$t_1^g (\Delta_A, 1) > t_2^g (\Delta_A, 1) \forall \Delta_A \in [-1, 1], j = \{g, n\} \Leftrightarrow \max \left\{ \tilde{\alpha}^g_A (1), \tilde{\alpha}^n_A (1) \right\} < -1 \Leftrightarrow \begin{cases} \bar{\alpha}_1 < \bar{\alpha}_2 + \beta_1 - \beta_2 \\ \bar{\alpha}_1 < \bar{\alpha}_2 + \beta_1 - \beta_2 \end{cases}$$

$$\tilde{\alpha}^n_A (1) = \frac{\alpha_2 - \alpha_1}{\beta_1 - \beta_2}, \tilde{\alpha}^g_A (1) = \frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2}$$

To make the analytical solution tractable, I fix the following parametrization:

$$\bar{\alpha}_1 = 0.5, \bar{\alpha}_2 = 0.1, \beta_1 = 2, \beta_2 = 1$$

and leave $\alpha_1$, $\alpha_2$ and $k$ free.

Since $\alpha_i$ has two possible realizations, $P (t_i^g)$ must depend on $\Delta_A$:

$$\Delta_A \in [0, 1] \Rightarrow t_i^g = \bar{\alpha}_i \Delta_A + \beta_i, P (t_i^g) = k$$

$$\Delta_A \in [-1, 0] \Rightarrow t_i^g = \bar{\alpha}_i \Delta_A + \beta_i, P (t_i^g) = 1 - k$$

and $J (\cdot)$ changes when passes $\Delta_A = 0$.

There are two equilibrium configurations that support $t_1^g (\Delta_A, 1) > t_2^g (\Delta_A, 1), j = \{g, n\}$ (all notations are specified in the proof of Proposition 4):

1. $E_1$ with $P (t_i^g) \in [0, \tilde{T} (\Delta_A, \Delta_B))$:
   - $t_1^g$ randomizes uniformly on $[0, T_1]$ and $(T_1, T_2]$ with probabilities $p (t_1^g)$ and $1 - p (t_1^g)$, respectively, $p (t_1^g) = \frac{P (t_1^g) (t_2^g - t_1^g) t_1^g}{t_1^g t_2^g P (t_1^g)}$;
   - $t_2^g$ uses a strategy including uniform randomization on $[0, T_1]$ and the atom $p_1^0 (t_2^g) \in (0, 1)$ at $e = 0$;
• $t_1^n$ randomizes uniformly on $(T_2, T_3)$ with probability 1;
• $t_2^n$ randomizes uniformly on $(T_1, T_2)$ and $(T_2, T_3)$ with probabilities $p(t_2^n)$ and $1 - p(t_2^n)$, respectively, $p(t_2^n) = \frac{t_2^n}{t_1^n}$.

2. $E_2$ with $P(t_2^n) \in \left[\tilde{T}(\triangle_A, \triangle_B), 1\right]:$
• $t_1^n$ randomizes uniformly on $[0, D_1]$ with probability 1;
• $t_2^n$ places the atom $p_3^n(t_j^n) = 1$ at $e = 0$;
• $t_2^n$ randomizes uniformly on $(D_1, D_2)$ with probability 1;
• $t_2^n$ randomizes uniformly on $[0, D_1]$ and $(D_1, D_2)$ with probabilities $p_1(t_2^n)$ and $p_2(t_2^n)$ respectively, and places the atom $\{1 - p_1(t_2^n) - p_2(t_2^n)\}$ at $e = 0$, $p_1(t_2^n) = \frac{p(t_2^n)}{p(t_2^n) + t_1^n}$.

First, take $\triangle_A \in [0, 1]$, i.e. $P(t_2^n) = k$. Also suppose $k \in \left[0, \tilde{T}(\triangle_A, 1)\right]$ for any $\triangle_A \in [0, 1]$, and the equilibrium configuration $E_1$ realizes. The “Winner–Takes–All” schedule is not optimal if $\frac{\partial J}{\partial \triangle_A} (1, 1) < 0$:

\[
\frac{\partial J}{\partial \triangle_A} (1, 1) < 0 \iff \gamma_1 k^2 + \gamma_2 k + \gamma_3 < 0
\]

where $\gamma_i, i = \{1, 2, 3\}$ are functions of $\alpha_1$ and $\alpha_2$ that can take both negative and positive values.

**Lemma 1.** There exist positive $\tilde{\alpha}_2(\alpha_1)$ (the function of $\alpha_1$) and $\tilde{k}$ such that $\frac{\partial J}{\partial \triangle_A} (1, 1) < 0$ holds for any $\alpha_2 \in [0, \tilde{\alpha}_2(\alpha_1))$ and $k \in \left[0, \tilde{k}\right]$.

**Proof.** Consider $\gamma_i, i = \{1, 2, 3\}$ as the functions of $\alpha_2$. Then $\gamma_1$ looks as follows:

\[
\gamma_1 = \delta_1^\gamma \alpha_2^2 + \delta_2^\gamma \alpha_2 + \delta_3^\gamma \\
\delta_1^\gamma = \sum_{j=0}^3 c_{1j} \omega_1^j, \delta_2^\gamma = \sum_{j=0}^3 c_{2j} \omega_1^j, l = \{2, 3\} \\
c_{1j}, c_{3j} > 0, c_{2j} < 0 \forall j
\]

where $\gamma_1 = 0$ always has two positive real roots, $u_1^\gamma$ and $u_2^\gamma$, $u_1^\gamma < u_2^\gamma$ such that $u_2^\gamma > \alpha_1$ and $u_1^\gamma > \alpha_1$ for any $\alpha_1 \in [0, \bar{\alpha}_1)$ and $\alpha_1 \in (0.12, \bar{\alpha}_1)$, respectively. Hence, under $\alpha_1 \in (0.12, \bar{\alpha}_1)$ the condition $\alpha_1 > \alpha_2$ implies $\gamma_1 > 0$.

Analyzing $\gamma_2$ and $\gamma_3$ delivers:

\[
\gamma_2 = \mu_1^\gamma \alpha_2^2 + \mu_2^\gamma \alpha_2 + \mu_3^\gamma \\
\mu_1^\gamma = \sum_{j=0}^2 d_{1j} \omega_1^j, \mu_2^\gamma = \sum_{j=0}^2 d_{2j} \omega_1^j, \mu_3^\gamma = \sum_{j=0}^2 d_{3j} \omega_1^j, d_{1j}, d_{2j} < 0, d_{3j} > 0 \forall j
\]

\[
\gamma_3 = \varphi_1^\gamma \alpha_2^2 + \varphi_2^\gamma \alpha_2 + \varphi_3^\gamma \\
\varphi_1^\gamma = \sum_{j=0}^1 s_{1j} \omega_1^j, \varphi_2^\gamma = \sum_{j=0}^2 s_{2j} \omega_1^j, \varphi_3^\gamma = s_3 \omega_1^j, s_{1j}, s_{2j} > 0, s_3 < 0 \forall j
\]
where \( \gamma_2 > 0 \) \((\gamma_3 > 0)\) holds under \( \alpha_2 \in [0, u_{\gamma_2}) \) \((\alpha_2 \in [u_{\gamma_3}, \min \{\alpha_1, \tilde{\alpha}_2\})\) and \( u_{\gamma_2} \) \((u_{\gamma_3})\) corresponds to the positive real root of \( \gamma_2 = 0 \) \((\gamma_3 = 0)\). Finally, \( u_{\gamma_2} > \alpha_1 \) is satisfied with \( \alpha_1 \in [0, 0.003) \) and \( u_{\gamma_3} < \alpha_1 \) for any feasible \( \alpha_1 \).

Further I investigate how \( u_{\gamma_2} \) and \( u_{\gamma_3} \) relate to each other. Both functions monotonically increase in \( \alpha_1 \), but \( u_{\gamma_2} \) changes faster.\(^{89} \) As a result, it must be \( u_{\gamma_2} > u_{\gamma_3} \), and for \( \alpha_1 \in (0.12, \bar{\alpha}_1) \) one can sign \( \gamma_i, i = \{1, 2, 3\} \) as follows:

\[
\alpha_2 \in [0, u_{\gamma_3}) \Rightarrow \gamma_{1,2} > 0, \gamma_3 < 0 \\
\alpha_2 \in [u_{\gamma_3}, u_{\gamma_2}) \Rightarrow \gamma_l > 0, l = \{1, 2, 3\} \\
\alpha_2 \in [u_{\gamma_2}, \min \{\alpha_1, \tilde{\alpha}_2\}) \Rightarrow \gamma_{1,3} > 0, \gamma_2 < 0
\]

Define \( \tilde{\alpha}_2 (\alpha_1) = u_{\gamma_3} \) and take \( \alpha_2 \in [0, \tilde{\alpha}_2 (\alpha_1)) \). Then equation \( \gamma_1 k^2 + \gamma_2 k + \gamma_3 = 0 \) has one positive real root \( \hat{k} \) such that \( \gamma_1 k^2 + \gamma_2 k + \gamma_3 < 0 \) for \( k \in [0, \hat{k}) \). Since the equilibrium structure \( E_1 \) requires \( k \in [0, \hat{T} (1, 1)] \), imposing \( \hat{k} = \min \{\hat{k}, \hat{T} (1, 1)\} \) gives the lemma.

Using the parameter sets specified in \textbf{Lemma 1}, I fix \( \alpha_1 = 0.15 \) and \( \alpha_2 = 0.01 \) but leave \( k \) free. Since \( \hat{T} (1, 1) \) exceeds \( \hat{k} \) under imposed restrictions, it must be \( \hat{k} = \hat{k} = 0.12 \).

\textbf{Lemma 2.} \( \hat{T} (\Delta_A, 1) \) decreases in \( \Delta_A \) for any \( \Delta_A \in [0, 1] \).

\textit{Proof.} \( \frac{\partial \hat{T} (\Delta_A, 1)}{\partial \Delta_A} > 0 \) holds iff \( \Delta_A \in (-\infty, -6.6) \cup (13.8) \) that has an empty intersection with \( \Delta_A \in [0, 1] \).\(^{90} \) Then \( \frac{\partial \hat{T} (\Delta_A, 1)}{\partial \Delta_A} < 0 \) for any \( \Delta_A \in [0, 1] \) follows. \( \square \)

\textbf{Lemma 2} implies that \( \hat{T} (\Delta_A, 1) \) has its maximum at \( \Delta_A = 0 \), and \( \hat{T} (0, 1) = \frac{1}{2} \). As a result, \( k \in [0, \hat{T} (1, 1)] \) is the strictest target, and no additional constraint on \( k \) is needed when \( \Delta_A \in [0, 1] \).

Further, I check how \( \frac{\partial \hat{J}}{\partial \Delta_A} (\Delta_A, 1) \) behaves with respect to \( \Delta_A \in [0, 1] \).

\textbf{Lemma 3.} \( \frac{\partial \hat{J}}{\partial \Delta_A} (\Delta_A, 1) \) increases in \( \Delta_A \in [0, 1] \).

\textit{Proof.} \( \frac{\partial \hat{J}}{\partial \Delta_A} (\Delta_A, 1) \) is non-negative iff: \(^{89} \)Derivatives of \( u_{\gamma_2} \) and \( u_{\gamma_3} \) with respect to \( \alpha_1 \) look as follows:

\[
\frac{\partial u_{\gamma_2}}{\partial \alpha_1} > 0 \forall \alpha_1 \in [0, 4.79) \\
\frac{\partial u_{\gamma_3}}{\partial \alpha_1} > 0 \forall \alpha_1 \in [0, 2.9)
\]

that implies \( \frac{\partial u_{\gamma_2}}{\partial \alpha_1} > 0, l = \{2, 3\} \) for any feasible \( \alpha_1 \) (remember \( \alpha_1 < \bar{\alpha}_1 = 0.5 \)). Also \( \frac{\partial u_{\gamma_2}}{\partial \alpha_1} > \frac{\partial u_{\gamma_3}}{\partial \alpha_1} \) holds under \( \alpha_1 \in [0, 7.84) \), and it must be \( \frac{\partial u_{\gamma_2}}{\partial \alpha_1} > \frac{\partial u_{\gamma_3}}{\partial \alpha_1} \) for any feasible \( \alpha_1 \) follows. \( \hat{J} \)

\(^{90} \)A complete derivative is \( \frac{\partial \hat{J} (\Delta_A, 1)}{\partial \Delta_A} = \frac{-5 \sum_{i=0}^{3} w_i \Delta_A^i}{[z (\Delta_A)]^2} \) where \( w_i < 0, w_{2,3} > 0 \) and \( z (\Delta_A) \) is a function of \( \Delta_A \).
\[\pi_1 k^2 + \pi_2 k + \pi_3 \geq 0\]

\[\pi_l = \sum_{j=0}^{8} q_{lj} \Delta_j^l, \quad l = \{1, 2, 3\}, \quad q_{1j}, q_{3j} > 0 \forall j\]

\[\pi_2 > 0 \forall \Delta_A \in [0, 1]\]

and the inequality holds for any \(k \geq 0\).\(^{91}\) As a result, \(\frac{\partial J}{\partial \Delta_A} (\Delta_A, 1)\) must increase in \(\Delta_A \in [0, 1]\).

Next, consider \(\Delta_A \in [-1, 0]\). Since \(P(t^*_1)\) and contestants’ types switch at \(\Delta_A = 0\), the equilibrium configuration in the interval of interest can differ. Suppose \(E_1\) is played in \(\Delta_A \in [-1, 0]\). Then \(P(t^*_1) = 1 - k\) must belong to \([0, \tilde{T}_s (\Delta_A, \triangle_B)]\) where \(\tilde{T}_s (\cdot)\) reflects the probability threshold in the case of changed types. Rewritten in terms of \(k\), the condition becomes \(k \in \left(1 - \tilde{T}_s (\cdot), 1\right]\).

**Lemma 4.** \(\tilde{T}_s (\Delta_A, 1)\) decreases in \(\Delta_A\) for any \(\Delta_A \in [-1, 0]\).

**Proof.** \(\frac{\partial \tilde{T}_s (\Delta_A, 1)}{\partial \Delta_A} > 0\) holds iff \(\Delta_A \in (-\infty, -19.8) \cup (-5.7)\) that has a non-empty intersection with \(\Delta_A \in (-1, 0]\).\(^{92}\) Then \(\frac{\partial \tilde{T}_s (\Delta_A, 1)}{\partial \Delta_A} < 0\) for any \(\Delta_A \in [-1, 0]\) follows.\(^{\square}\)

Given **Lemma 4,** \(\left\{1 - \tilde{T}_s (\cdot)\right\}\) must increase in \(\Delta_A \in [-1, 0]\):

\[
\sup \left\{1 - \tilde{T}_s (\Delta_A, 1)\right\} = 1 - \tilde{T}_s (0, 1) = \frac{1}{2}
\]

\[
\inf \left\{1 - \tilde{T}_s (\Delta_A, 1)\right\} = 1 - \tilde{T}_s (-1, 1) = 0.41
\]

However, the sets \(k \in \left(1 - \tilde{T}_s (0, 1), 1\right]\) and \(k \in [0, \bar{k}]\) have an empty intersection. Hence, \(\frac{\partial J}{\partial \Delta_A} (1, 1) < 0\) and \(E_1\)-type equilibrium in \(\Delta_A \in [-1, 0]\) result in a contradiction.

Now assume \(E_2\) is played in \(\Delta_A \in [-1, 0]\). Then the relevant equilibrium constrain on \(k\) becomes \(k \in [0, 1 - \tilde{T}_s (\cdot)]\), and this is implied by \(k \in [0, \bar{k}]\):

\[
\inf \left\{1 - \tilde{T}_s (\Delta_A, 1)\right\} > \bar{k}
\]

Let \(\frac{\partial J}{\partial \Delta_A} (\Delta_A, 1)\) be the derivative of \(J (\cdot)\) with respect to \(\Delta_A \in [-1, 0]\).

**Lemma 5.** \(\frac{\partial J}{\partial \Delta_A} (\Delta_A, 1)\) increases in \(\Delta_A \in [-1, 0]\).

**Proof.** \(\frac{\partial^2 J}{\partial \Delta_A^2} (\Delta_A, 1)\) looks as follows:

\[
\frac{\partial^2 J}{\partial \Delta_A^2} (\Delta_A, 1) = \frac{\sum_{i=0}^{3} y_{1i} \Delta_i^2 + \sum_{i=0}^{3} y_{2i} \Delta_i^2}{(\sum_{j=0}^{2} y_{j0} \Delta_j^1)^2}
\]

\[y_{2j}, y_{3j} > 0 \forall \Delta_A, y_{11} < 0\]

\(^{91}\)When \(\Delta_A \geq 0\), positive entries of \(\pi_2\) over-compensate negative ones.

\(^{92}\)A complete derivative is \(\frac{\partial \tilde{T}_s (\Delta_A, 1)}{\partial \Delta_A} = \frac{-40 \sum_{j=0}^{2} w_{s j} \Delta_j^1}{(z_s (\Delta_A))^2}\) where \(w_{sj} > 0 \forall j\) and \(z_s (\Delta_A)\) is a function of \(\Delta_A\).
where polynomial coefficients are positive under \( \triangle_A \in [-1, 0] \).\(^{93}\) Hence, \( \frac{\partial J}{\partial \triangle_A} (\triangle_A, 1) \) must increase in \( \triangle_A \in [-1, 0] \) for any feasible \( k \).

\[ \square \]

Since \( \frac{\partial J}{\partial \triangle_A} (\triangle_A, 1) \) is continuous (no switching points were assumed) and increasing in \( \triangle_A \in [-1, 0] \), the condition \( \frac{\partial J}{\partial \triangle_A} (0, 1) < 0 \) is sufficient for \( J(\triangle_A, 1) \) to decrease in \( \triangle_A \in [-1, 1] \).\(^{94}\)

\[ \frac{\partial J}{\partial \triangle_A} (0, 1) = -m_1k^2 - m_2, m_i > 0, i = \{1, 2\} \]

As a result, for \( k \in \left[0, \tilde{k}\right] \) the designer’s objective \( J(\triangle_A, 1) \) decreases in \( \triangle_A \in [-1, 1] \). Thus, it is optimal to choose \( \triangle_A = -1 \) given \( \triangle_B = 1 \) and use good \( A \)’s endowment completely.

Further I verify the optimality of \( \triangle_B = 1 \). The proposed valuation structure features feasible identity switching and cutoff points:

1. \( \triangle_A \geq 0 \): type \( t_i^n \) is well-defined only for \( \triangle_B \in \left[ -\frac{\alpha_2}{\beta_2} \triangle_A, 1 \right] \), \( \left\{ -\frac{\alpha_1}{\beta_1} \triangle_A \right\} \in (-1, 0) \), and no switch in contestants’ power takes place:

\[ \tilde{\Delta}_B^g (\triangle_A) < \tilde{\Delta}_B^n (\triangle_A) = \frac{\alpha_2 - \alpha_1}{\beta_1 - \beta_2} \Delta_A < -\frac{\alpha_2}{\beta_2} \Delta_A \forall \Delta_A \geq 0 \]

2. \( \triangle_A < 0 \): type \( t_i^n \) is well-defined only for \( \triangle_B \in \left[ -\frac{\alpha_1}{\beta_1} \triangle_A, 1 \right] \), \( \left\{ -\frac{\alpha_1}{\beta_1} \triangle_A \right\} \in (0, 1) \), and the power of \( n \) types switches on the interval:

\[ \tilde{\Delta}_B^g (\triangle_A) < -\frac{\alpha_1}{\beta_1} \Delta_A < \tilde{\Delta}_B^n (\triangle_A) = \frac{\alpha_2 - \alpha_1}{\beta_1 - \beta_2} \Delta_A \forall \Delta_A < 0 \]

and for \( \triangle_B \in \left[ -\frac{\alpha_1}{\beta_1} \triangle_A, \tilde{\Delta}_B^n (\triangle_A) \right] \) type \( n \) of contestant 2 has stronger incentives to win.

Since \( \beta_i \) is deterministic, \( P(t_i^n) \) does not change when \( J(\triangle_A, \triangle_B) \) passes \( \Delta_B = 0 \) for given \( \triangle_A \). First, take the case of \( \Delta_A \geq 0 \) when \( E_1 \) is played. Let \( \frac{\partial J}{\partial \Delta_B} (\triangle_A^+, \Delta_B) \) be the derivative of \( J(\triangle_A^+) \) with respect to \( \Delta_B \) when \( \Delta_A \geq 0 \).

**Lemma 6.** There exists \( \tilde{k} \in [0, 1] \) such that for any \( k \in [0, \tilde{k}] \) the derivative \( \frac{\partial J}{\partial \Delta_B} (\triangle_A^+, \Delta_B) \) increases in \( \Delta_B \), \( \Delta_B \in \left[ -\frac{\alpha_2}{\beta_2} \triangle_A^+, 1, 1 \right] \).

**Proof.** \( \frac{\partial^2 J}{\partial \Delta_B^2} (\triangle_A^+, \Delta_B) \) looks as follows:

\[ \frac{\partial^2 J}{\partial \Delta_B^2} (\triangle_A^+, \Delta_B) = \frac{\nu_i k^2 + \nu_j k + \nu_3}{(3 \Delta_A + 40 \Delta_B)^3} \]

\[ \nu_i = \sum_{j=0}^{8} h_{ij} \Delta_A^j \Delta_B^{8-j}, i = \{1, 2, 3\} \]

\[ h_{ij}, h_{3j}, \Delta_A > 0 \forall j \]

\(^{93}\)\( \sum_{j=0}^{3} y_{1j} \Delta_A^j \) has a single real root \( \Delta_A^1 = 19.6 \) such that \( \sum_{j=0}^{3} y_{1j} \Delta_A^j > 0 \) for any \( \Delta_A < \Delta_A^1 \). Coefficient \( \sum_{j=0}^{3} y_{2j} \Delta_A^j \) reduces to \( (\Delta_A + 4)^3 \) and is always positive under \( \Delta_A \in [-1, 0] \). Polynomial \( \sum_{j=0}^{2} y_{3j} \Delta_A^j \) has two negative real roots located below \( \Delta_A = -1 \).

\(^{94}\)Continuity of \( \frac{\partial J}{\partial \Delta_A} (\Delta_A, 1) \) and \( \frac{\partial J}{\partial \Delta_A} (0, 1) < 0 \) imply \( \frac{\partial J}{\partial \Delta_A} (\Delta_A, 1) < 0 \) in a neighborhood to the left of \( \Delta_A = 0 \).
Since \( \Delta_B \) cannot be lower than \( \left\{ -\frac{\alpha_2}{\beta_2} \Delta_A^+, 1 \right\} \), the denominator of this expression is always positive. Moreover, both \( \nu_1 > 0 \) and \( \nu_3 > 0 \) hold for any feasible \( \Delta_B \),\(^{95}\) but the sign of \( \nu_2 \) is ambiguous. If \( \nu_2 > 0 \) or \( \nu_2 < 0 \), but the discriminant of \( \nu_1 k^2 + \nu_2 k + \nu_3 = 0 \) is negative, any \( k \in [0, 1] \) results in \( \frac{\partial J}{\partial \Delta_B} \left( \Delta_A^+, \Delta_B \right) > 0 \), and one can take \( \bar{k} = 1 \). Otherwise, \( \nu_2 < 0 \) combined with the positive discriminant delivers two real roots of \( \nu_1 k^2 + \nu_2 k + \nu_3 = 0 \), \( k_1 \) and \( k_2 \), \( k_1 < k_2 \), such that \( \frac{\partial J}{\partial \Delta_B} \left( \Delta_A^+, \Delta_B \right) \geq 0 \) holds for \( k \in [0, k_1] \cup [k_2, \infty) \). Then taking \( \bar{k} = \min \{1, k_1\} \) gives the lemma.

\[ \square \]

Using Lemma 6, I restrict \( k \in \left[0, \min \{\bar{k}, \hat{k}\}\right] \). Given that \( \frac{\partial J}{\partial \Delta_B} \left( \Delta_A^+, \Delta_B \right) \) is continuous and increasing in \( \Delta_B \in \left[-\frac{\alpha_2}{\beta_2} \Delta_A^+, 1\right] \), the condition \( \frac{\partial J}{\partial \Delta_B} \left( \Delta_A^+, -\frac{\alpha_2}{\beta_2} \Delta_A^+ \right) > 0 \) is sufficient to guarantee \( \frac{\partial J}{\partial \Delta_B} \left( \Delta_A^+, \Delta_B \right) > 0 \) for any \( \Delta_A^+ \) and feasible \( \Delta_B \):

\[ \frac{\partial J}{\partial \Delta_B} \left( \Delta_A^+, \Delta_B \right) > 0 \iff \tau_1 k^2 + \tau_2 k + \tau_3 > 0 \]

where \( \tau_1 < 0 \), \( \tau_2, \tau_3 > 0 \). The inequality holds for \( k \in (-0.4, 2.5) \), and this set includes all feasible values of \( k \). Hence, \( \frac{\partial J}{\partial \Delta_B} \left( \Delta_A^+, \Delta_B \right) \) is positive for any \( \Delta_A^+, \Delta_B \in \left[-\frac{\alpha_2}{\beta_2} \Delta_A^+, 1\right] \) and \( k \in \left[0, \min \{\bar{k}, \hat{k}\}\right] \).

Next, I analyze the case of \( \Delta_A < 0 \) where \( E_2 \) is played. Define \( \frac{\partial J}{\partial \Delta_B} \left( \Delta_A^-, \Delta_B \right) \) as the derivative of \( J(\cdot) \) with respect to \( \Delta_B \). Since type \( n \) of contestant 2 has stronger incentives to win when \( \Delta_B \in \left[-\frac{\alpha_1}{\beta_1} \Delta_A^-, \bar{\Delta}_B^n \left( \Delta_A^- \right) \right] \), one must consider this interval and \( \Delta_B \in \left[\bar{\Delta}_B^n \left( \Delta_A^- \right), 1\right] \) separately. Let \( E_2^+ \) and \( E_2^- \left( \frac{\partial J}{\partial \Delta_B} \left( \Delta_A^-, \Delta_B \right) \right) \) and \( \frac{\partial J}{\partial \Delta_B} \left( \Delta_A^-, \Delta_B \right) \) correspond to the equilibria (the derivatives) to the right and to the left of \( \bar{\Delta}_B^n \left( \Delta_A^- \right) \), respectively. Define \( \tilde{T}_s (\Delta_A^-, \Delta_B) \) as the probability threshold for \( \Delta_A^- \) and \( \Delta_B \in \left[-\frac{\alpha_1}{\beta_1} \Delta_A^-, \bar{\Delta}_B^n \left( \Delta_A^- \right) \right] \).

Lemma 7. \( \tilde{T}_s (\Delta_A^-, \Delta_B) \) and \( \tilde{T}_s (\Delta_A^-, \Delta_B) \) decrease in \( \Delta_B \) for any \( \Delta_B \in \left[\bar{\Delta}_B^n \left( \Delta_A^- \right), 1\right] \) and \( \Delta_B \in \left[-\frac{\alpha_1}{\beta_1} \Delta_A^-, \bar{\Delta}_B^n \left( \Delta_A^- \right) \right] \), respectively.

Proof. \( \frac{\partial \tilde{T}_s \left( \Delta_A^-, \Delta_B \right)}{\partial \Delta_B} \) and \( \frac{\partial \tilde{T}_r \left( \Delta_A^+, \Delta_B \right)}{\partial \Delta_B} \) look as follows:

\[
\frac{\partial \tilde{T}_s \left( \Delta_A^-, \Delta_B \right)}{\partial \Delta_B} = \frac{\varphi_1 \left( \Delta_A^-, \Delta_B \right)}{\left[ \varphi_1 \left( \Delta_A^-, \Delta_B \right) \right]^2}
\]

\[
\frac{\partial \tilde{T}_r \left( \Delta_A^+, \Delta_B \right)}{\partial \Delta_B} = \frac{\varphi_1 \left( \Delta_A^+, \Delta_B \right)}{\left[ \varphi_2 \left( \Delta_A^+, \Delta_B \right) \right]^2}
\]

\[
\varphi_1 \left( \Delta_A^-, \Delta_B \right) = \sum_{j=0}^{2} y_{ij} \Delta_A^+ \Delta_B^{2-j}, \quad l = \{1, 2\}
\]

where \( y_{ij} > 0 \) for any \( l, j \) and \( x_l \left( \Delta_A^-, \Delta_B \right) \) is the function of \( \Delta_A^-, \Delta_B \). Solving \( \varphi_1 \left( \Delta_A^-, \Delta_B \right) \) with

\(^{95}\)When \( \Delta_B \geq 0 \), all elements of the polynomials exceed zero. When \( \Delta_B < 0 \), positive entries over-compensate negative ones corresponding to odd powers.
respect to $\Delta_B$ brings that $\varphi_*(\Delta_A, \Delta_B)$ is positive for any $\Delta_B > -0.17\Delta_A$, and this includes all feasible values of $\Delta_B$. Similarly, $\varphi_1(\Delta^-_A, \Delta_B) > 0$ holds under $\Delta_B > -0.11\Delta_A$ that covers $\Delta_B \in \left[ -\frac{\alpha_1}{\beta_1}\Delta_A, \tilde{\Delta}_B^n(\Delta^-_A) \right]$. Thus, for $\Delta^-_A \in [-1, 0)$ it must be $\frac{\partial \varphi_*(\Delta_A, \Delta_B)}{\partial \Delta_B} < 0$ and $\frac{\partial \varphi_1(\Delta_A, \Delta_B)}{\partial \Delta_B} < 0$ for any $\Delta_B \in \left[ \tilde{\Delta}_B^n(\Delta^-_A), 1 \right]$ and $\Delta_B \in \left[ -\frac{\alpha_1}{\beta_1}\Delta_A, \tilde{\Delta}_B^n(\Delta^-_A) \right]$, respectively.

\[ \square \]

**Lemma 7** implies that \( \left\{ 1 - \tilde{T}_s(\Delta^-_A, \Delta_B) \right\} \) and \( \left\{ 1 - \tilde{T}^*_s(\Delta^-_A, \Delta_B) \right\} \) must increase in $\Delta_B$ and have infima at $\Delta_B = \tilde{\Delta}_B^n(\Delta^-_A)$ and $\Delta_B = -\frac{\alpha_1}{\beta_1}\Delta_A$, respectively:

\[
\begin{align*}
\inf \left\{ 1 - \tilde{T}_s(\Delta_A, \Delta_B) \right\} &= 0.24 \\
\inf \left\{ 1 - \tilde{T}^*_s(\Delta_A, \Delta_B) \right\} &= 0.16
\end{align*}
\]

Given that $\tilde{k}$ is smaller than any of these values, $k \in \left[ 0, \min \left\{ \tilde{k}, \tilde{k}\right\} \right]$ implies both $k \in \left[ 0, 1 - \tilde{T}_s(\Delta^-_A, \Delta_B) \right]$ and $k \in \left[ 0, 1 - \tilde{T}^*_s(\Delta^-_A, \Delta_B) \right]$. Hence, no additional restrictions are needed on $k$.

**Lemma 8.** $\frac{\partial J^+}{\partial \Delta_B}(\Delta^-_A, \Delta_B)$ and $\frac{\partial J^-}{\partial \Delta_B}(\Delta^-_A, \Delta_B)$ increase in $\Delta_B$ for any feasible $\Delta_B$ and $\Delta^-_A$.

**Proof.** Consider $\frac{\partial^2 J}{\partial \Delta_B^2}(\Delta^-_A, \Delta_B)$:

$$
\frac{\partial^2 J}{\partial \Delta_B^2}(\Delta^-_A, \Delta_B) = \frac{\rho_1 k^2 + (\Delta_A + 10\Delta_B)^3}{(3\Delta_A + 40\Delta_B)^4(\Delta_A + 10\Delta_B)^4}
$$

$$
\rho_1 = \sum_{j=0}^{3} x_{1j} \Delta_A^3 \Delta_B^{3-j}, x_{1j} > 0 \forall j
$$

The denominator and the second component of the numerator increase in $\Delta_B$, are positive at $\Delta_B = \Delta_B^{min}$, $\Delta_B^{min} = \left\{ -\frac{\alpha_1}{\beta_1}\Delta_A \right\}$ and, consequently, any feasible $\Delta_B$. Consider $\rho_1$ as the function of $\Delta_B$ (recall $\Delta_A < 0$):

$$
\frac{\partial \rho_1(\Delta_B)}{\partial \Delta_B} \leq 0 \iff \Delta_B \in [-0.05, -0.08] \Delta^-_A
$$

The set has an empty intersection with $\Delta_B \in \left[ -\frac{\alpha_1}{\beta_1}\Delta_A, \tilde{\Delta}_B^n(\Delta^-_A) \right]$, and $\rho_1$ must increase in $\Delta_B$ for any feasible $\Delta_B$. Then estimate $\rho_1(\Delta_B^{min})$:

$$
\rho_1(\Delta_B^{min}) = - (\Delta_A)^3 (p_1 k^2 + p_2) > 0, p_{1,2} > 0
$$

This implies $\rho_1(\Delta_B) > 0$ for any feasible $\Delta_B$. As a result, $\frac{\partial J}{\partial \Delta_B}(\Delta^-_A, \Delta_B)$ must increase in $\Delta_B$ for any feasible $\Delta_B$ and $\Delta^-_A$.

Next, I investigate $\frac{\partial^2 J^+}{\partial \Delta_B^2}(\Delta^-_A, \Delta_B)$:

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\[
\frac{\partial^2 J^{+}}{\partial \Delta_B} (\Delta_A, \Delta_B) = \frac{\rho_2 k^2 + (\Delta_A + 4\Delta_B)^3}{(3\Delta_A + 4\Delta_B)^3 (\Delta_A + 4\Delta_B)^3}
\]
\[\rho_2 = \sum_{j=0}^{3} x_{2j} \Delta_A^j \Delta_B^{3-j}, \quad x_{2j} > 0 \forall j\]

Again, the denominator and the second element of the numerator are positive for all \(\Delta_B \in \left[\Delta_B^u (\Delta_A), 1\right]\). Treating \(\rho_2\) as the function of \(\Delta_B\) delivers:

\[
\frac{\partial \rho_2(\Delta_B)}{\partial \Delta_B} \leq 0 \iff \Delta_B \in [-0.01, -0.11] \Delta_A
\]

\[
\rho_2(\Delta_B^{\text{min}}) = -p_3 (\Delta_A^{-})^3 > 0, \quad p_3 > 0
\]

and \(\rho_2\) must be positive for any feasible \(\Delta_B\). Hence, \(\frac{\partial^2 J^{+}}{\partial \Delta_B} (\Delta_A, \Delta_B) > 0\) follows, and \(\frac{\partial J^{+}}{\partial \Delta_B} (\Delta_A, \Delta_B)\) has to increase in \(\Delta_B\) for any feasible \(\Delta_B\) and \(\Delta_A\).

**Lemma 8** implies that a sufficient condition for \(\frac{\partial J}{\partial \Delta_B} (\Delta_A, \Delta_B)\) to increase in \(\Delta_B\) for any \(\Delta_A\) is \(\frac{\partial J}{\partial \Delta_B} \left( -\frac{\alpha_1}{\beta_1} \Delta_A, \Delta_B \right) > 0\) and \(\frac{\partial J^{+}}{\partial \Delta_B} \left( \Delta_B^u (\Delta_A), \Delta_B \right) > 0\): \[\Box\]

\[
\frac{\partial J^{+}}{\partial \Delta_B} \left( \Delta_B^u (\Delta_A), \Delta_B \right) = -(\Delta_A^{-})^4 (q_1 k^2 - q_2), \quad q_1, 2 > 0, \frac{q_2}{q_1} > 1
\]

\[
\frac{\partial J}{\partial \Delta_B} \left( -\frac{\alpha_1}{\beta_1} \Delta_A, \Delta_B \right) = (\Delta_A^{-})^4 (q_3 k^2 + q_4), \quad q_3, 4 > 0
\]

As a result, for \(k \in \left[0, \min \left\{ \tilde{k}, \tilde{k} \right\} \right]\) the objective function \(J (\Delta_A, \Delta_B)\) monotonically increases in \(\Delta_B\) for any feasible \(\Delta_A\), and \(\Delta_B = 1\) is indeed optimal.

To sum up, for the set of symmetric probability-valuation profiles \(\bar{R}_k^L = \{0.5, 0.1, 0.15, 0.01, 2, 1, k\} \in R_k^L\) such that \(k \in \left[0, \min \left\{ \tilde{k}, \tilde{k} \right\} \right]\) the designer chooses \(\Delta_A = -1, \Delta_B = 1\) and uses goods’ endowments completely. \[\Box\]