WORST-CASE APPROACH TO THE DISCRETE-TIME STOCHASTIC OPTIMAL CONTROL WITH APPLICATION TO THE PORTFOLIO SELECTION PROBLEM [1]

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Stochastic programming approach to optimal portfolio selection is widely used in academic literature since the pioneering works of Merton [Merton, 1969] and Samuelson [Samuelson, 1969] who studied the problem for discrete and continuous time in its simplest form (multi-asset portfolio on a zero-cost market without price impact). The approach has been extended in various studies, see [Xu and Shreve, 1992] for an overview. Extensive research has been conducted recently for market with transaction costs and price impact. Still, most of the research in this area is centered around continuous time market where prices follow the geometric Brownian motion, e. g. [Fruth et al., 2013, Obizhaeva and Wang, 2013]. The model itself has been criticized due to its underestimation of the probability of large price movements [Mantegna and Stanley, 1999, Cont, 2001], which is crucial during market shocks and high volatility.

This paper presents an alternative statement of the portfolio selection problem in a discrete time market. The statement does not require specification of the stochastic dynamics of the system. Instead, basic properties of the system parameter conditional distribution must be specified (the expected value and range) for the remaining investment periods. Characteristics of the distribution can be estimated from the observable statistics or given by an expert. Optimal strategy is assumed to maximize the worst-case expected value of the general-shaped terminal utility function. The approach admits transaction costs and phase constraints while being oriented for practical use as a decision support system (DSS) during an investment management process. The proposed framework is based on the robust dynamic programming approach, for further references see [Nilim and El Ghaoui, 2005] for finite state space and finite decision choices at each moment, or [Iyengar, 2005] for infinitely countable spaces. We consider a general case of uncountable space of states and strategies but impose a requirement of compactness and convexity of the parameter domain. We also consider the Markovian market dynamics, which is similar to the “rectangularity” assumption of [Iyengar, 2005]. In finance, a similar approach has been studied in [Deng et al., 2005] for

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a one-period problem and a Markowitz optimal criterion without transaction costs. The worst-case framework has also been recently presented in [Chigodaev, 2016] for the option super-hedging problem on a zero-cost market with one risky asset, based on the ideas of Prof. Smirnov introduced at the course for the Moscow State University. For an overview of other robust optimization techniques, see [Bertsimas et al., 2011].

1 Worst-case portfolio optimization

Consider a discrete market of m risky assets and one risk-free asset at moments $t_0, \dots, t_N$ with a standard filtration $\{\mathcal{F}_n\}_{n=0}^N$. The stochastic market parameters are known to follow the Markovian $\mathcal{F}_n$-measurable process Θ_n with a distribution Q from the class of distributions $\mathbb{Q}^E$ with the expected value E_n and compact support K_n at t_n, $n = \overline{1, N}$. Unlike the classic stochastic programming approach, the distribution Q, hence the market model, is ambiguous. Let $\mathcal{S}$ denote the state of the whole dynamic system, i. e. state of the market and the portfolio. The initial state $\mathcal{S}_0$ is given.

Let X_n and Y_n be the risky and the risk-free prices at all t_n respectively. By portfolio at t_n we mean a vector H_n of positions in risk-free and risky assets. We assume that there are no external flows of any assets so the budget equation allows us to express the risk-free position H_n^Y at t_n via the position in other assets H_n^X and the portfolio wealth W_{n-1} as

$$H_n^Y = Y_{n-1}^{-1} \left(W_{n-1} - {H_n^X}^T X_{n-1} - C_{n-1}(\Delta H_n^X, \mathcal{S}_{n-1}) \right), \tag{1}$$

where C_{n-1} is the estimated value of transaction costs for the deal of size ΔH_n^X performed at the n-th investment period given the state $\mathcal{S}_{n-1}$ at t_{n-1}.

By *strategy* we mean a sequence $H^X = \{H_k^X\}_{k=1}^N$ where H_n^X is $\mathcal{F}_{n-1}$-measurable. We also assume that the strategy can be subject to certain trading limits which are known to the portfolio manager at the time of the decision: $H_n^X \in D_{n-1}(\mathcal{S}_{n-1})$, where $D_{n-1}(\mathcal{S}_{n-1})$ is $\mathcal{F}_{n-1}$-measurable. The class of all admissible Markovian strategies is denoted $\mathcal{A}$.

Let $\mathcal{S} \mid H^X$ be the state when the portfolio's risky-asset position is H^X. By *optimal strategy* we mean $H^{X^*} \in \mathcal{A}$ such that

$$\sup_{H^X \in \mathcal{A}} \inf_Q \mathbb{E}_Q^{\mathcal{S}_0} J(S_N \mid H_N^X) = \inf_Q \mathbb{E}_Q^{\mathcal{S}_0} J(\mathcal{S}_N \mid H_N^{X^*}),$$

where $J(\mathcal{S})$ is a given utility function. Consider the ambiguous Markovian parameter process

Θ_n. Then the Bellman-Isaacs equation of the problem is

$$V_{n-1}(\mathcal{S}) = \sup_{H_n \in D_n(\mathcal{S})} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}^{\mathcal{S}}_{Q_n} V_n(\mathcal{S}_n \mid H_n), \quad \mathcal{S} \in \mathbb{S}_{n-1}, \qquad n = \overline{1, N}, \tag{2}$$

$$V_N(\mathcal{S}) = J(\mathcal{S}), \quad \mathcal{S} \in \mathbb{S}_N, \tag{3}$$

where $V_n(\mathcal{S})$ is the value function at t_n, Q_n is the marginal distribution of Θ_n and $\mathbb{Q}_n^E$ is the set of such distributions. The strategies satisfying

$$H_n^* \in \operatorname*{Arg\,max}_{H_n \in D_n(\mathcal{S})} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}^{\mathcal{S}}_{Q_n} V_n(\mathcal{S}_n \mid H_n), \qquad \mathcal{S} \in \mathbb{S}_{n-1}, \tag{4}$$

can be proven optimal via the corresponding Verification theorem.

To demonstrate the worst-case approach, consider the multiplicative dynamics of prices:

$$\Delta X_n = \Delta \xi_n^X X_{n-1}, \qquad \Delta Y_n = \Delta \xi_n^Y Y_{n-1}.$$

Consider the standard risk-free price dynamics with $\Delta \xi_n^Y = r_n \Delta t_n$, where r_n is the risk-free rate. Any $\Delta \xi_n^X$ can be represented as $\Delta \xi_n^X = \mu_n \Delta t_n + \sigma_n \sqrt{\Delta t_n}$, so the dynamics can be written as

$$\Delta X_n = \mu_n X_{n-1} \Delta t_n + \sigma_n X_{n-1} \sqrt{\Delta t_n},\ n = \overline{1, N}, \tag{5}$$

$$\Delta Y_n = r_n Y_{n-1} \Delta t_n,\ n = \overline{1, N}, \tag{6}$$

In the research, we assume that r_n and the drift μ_n are known while $\sigma_n \equiv \Theta_n \sim Q_n \in \mathbb{Q}_n^E$ is the only ambiguous parameter.

The main difficulty in finding a solution according to (4) is solving an internal minimization problem over a set of measures. We use results from the theory of Chebyshev's inequalities [Karlin and Studden, 1966] to show that if $V_n(\mathcal{S}_n \mid H_n)$ which depends on σ_n via $\mathcal{S}_n$, is in fact concave in σ_n, then the solution of the internal problem contains a discrete measure with the mass concentrated at $m+1$ or less boundary points of the compact support K_n. Furthermore, if K_n is a polyhedron, the measure is necessarily concentrated in its extreme points. Thus the problem of finding an optimal solution is reduced to the problem of finding sufficient conditions for σ-concavity of V_n defined by the Bellman-Isaacs equation (2)-(3).

We provide sufficient conditions for the concavity which affect the form of the terminal utility function, the transaction costs and the trading limit sets. Considering the Markovian nature of the market and the portfolio strategy, the system state can be represented as $\mathcal{S} = (X, H^X, W^Y)$, where X is the current risky price vector, H^X is the portfolio structure and W^Y is the value of the risk-free position (e. g. the available capital).

Let $\mathbb{S}^* \subseteq \mathbb{S}$. While solving the Bellman equation we usually assume that all the functions are defined for such states that do not lead to degenerate cases, i.e. empty set of admissible strategies ($D(\mathcal{S}) = \emptyset$).

Assumption 1. *$D(X, H^X, W^Y)$ is such that*

1. *For every $(X, H^X, W^Y) \in \mathbb{S}^*$*

$$Z \in D(X, H^X, W^Y) \quad \Longleftrightarrow \quad AZ \in D(A^{-1}X, AH^X, W^Y) \tag{7}$$

$$\forall A = \mathrm{diag}(a^1, \dots, a^m) > 0.$$

2. *For every $(X, H^X, W^Y) \in \mathbb{S}^*$*

$$Z \in D(X, H^X, W^Y) \quad \Longleftrightarrow \quad Z \in D(X, 0, W^Y + {H^X}^T X); \tag{8}$$

3. *For every $\alpha \in [0, 1]$ and every $(X, H_1^X, W_1^Y), (X, H_2^X, W_2^Y) \in \mathbb{S}^*$,*

$$Z_1 \in D(X, H_1^X, W_1^Y),\ Z_2 \in D(X, H_2^X, W_2^Y) \quad \Rightarrow$$

$$\Rightarrow \ \alpha Z_1 + (1 - \alpha) Z_2 \ \in \ D(X, \alpha H_1^X + (1 - \alpha) H_2^X, \alpha W_1^Y + (1 - \alpha) W_2^Y). \tag{9}$$

First part of the assumption reflects the idea that a position of v securities can be replaced by a position of $a \cdot v$ different securities if their price is known to be always a times less; the same is also true for the trades. This assumption holds in a frictionless market since the value of positions (trades) will be the same before and after transformation. In the presence of implicit transaction costs it might no longer be true if the costs depend on the volume of a trade and not its market value. The second part of the assumption states that trading limits are defined by the total market value of the portfolio rather than its structure. The third part represents a convexity requirement for the trading limit sets, in fact classic convexity of D follows from (9). Here is an example of the function which satisfies Assumption 1:

$$D(X, H^X, W^Y) = \left\{Z \in \mathbb{R}^m \colon -\beta^X W \le Z^T X \le (1+\beta^Y) W\right\}, \tag{10}$$

$$W = W^Y + {H^X}^T X.$$

The limits here are defined in terms of market value of positions as a percentage of the portfolio market value.

Now consider a function $V(X, H^X, W^Y) : \mathbb{S}^* \to \mathbb{R}$.

Assumption 2. *$V(X, H^X, W^Y)$ is such that for every $(X, H^X, W^Y) \in \mathbb{S}^*$*

$$V(AX, H^X, W^Y) = V(X, AH^X, W^Y) \quad \forall A = \operatorname{diag}(a^1, \dots, a^m) > 0. \tag{11}$$

Assumption 3. *$V(X, H^X, W^Y)$ is such that for every $(X, H^X, W^Y) \in \mathbb{S}^*$*

$$V(X, H^X, W^Y) = V(X, 0, W^Y + {H^X}^T X). \tag{12}$$

Assumption 2 is interpreted similarly to Assumption 1.1. It states that the expected payoff of the portfolio does not change if some of the risky assets would be split or merged. Assumption 3 is similar to Assumption 1.2 and holds when the payoff of the portfolio is defined by the value of positions rather than its structure. An example of the utility satisfying both Assumptions is any classic utility function depending solely on the portfolio market value (CRRA, CARA utilities).

We consider the following costs functions: $C_{n-1}(\Delta H^X, \mathcal{S}_{n-1}) \overset{\mathrm{df}}{=} C_{n-1}(\Delta H^X, X_{n-1})$, $n = \overline{1, N}$. While being quite narrow, this simple class includes the most commonly used proportional costs function.

Consider a function $C(\Delta H, X) : \mathbb{R}^m \times \mathbb{R}_+^m \to \mathbb{R}_+$.

Assumption 4. *$C(\Delta H, X)$ is such that*

1. *$C(\Delta H, X)$ is non-negative, non-decreasing in $|\Delta H|$ and convex in ΔH on $\mathbb{R}^m$ for any $X \in \mathbb{R}_+^m$;*

2. *For every* $X \in \mathbb{R}^m_+$, $\Delta H \in \mathbb{R}^m$ *and* $A = \operatorname{diag}(a^1, \dots, a^m) > 0$

$$C(A\Delta H, X) = C(\Delta H, AX); \tag{13}$$

First part of the Assumption is a set of common conditions for the cost function based on its economic meaning. For an order driven market, it can be proven that these properties are inherent to any implicit transaction costs function. Property (13) holds when costs are defined by the market value of the trades, e. g. equal a percentage of the deal value. Note that the property is true for fixed and proportional cost functions.

Now we can provide sufficient conditions for simplifying the Bellman-Isaacs equation so it can be solved numerically. In this research we only consider the case of polyhedral support K_n for the ambiguous distributions. For a bounded polyhedron K, let $m(K) = \dim(K)$ and let $\mathcal{G}(K, E)$ be a set of combinations of its $m(K) + 1$ extreme points such that their convex combination contains a point $E \in K$. For a combination $G = (G_0, \dots, G_{m(K)}) \in \mathcal{G}(K, E)$, let $p^i(G, E)$ be the barycentric coordinate of E in the convex combination corresponding to the point G_i, $i = \overline{0, m(K)}$. Also consider for every $n = \overline{1, N}$ a subset $\mathbb{S}^*_n \subseteq \mathbb{S}_n$ of states for which $D_n(\mathcal{S}) \neq \emptyset$ and well-defined, besides $J(\mathcal{S})$ is well-defined on $\mathbb{S}^*_N$. The structure of the $\mathbb{S}^*_n$ sequence is quite complicated, heavily depends on the particular statement of the problem and requires an additional research which is out of the scope of this paper. For additional results see [Andreev, 2017].

Theorem 1.1. *Let the following assumptions hold:*

1. $J(X, H^X, W^Y)$ *is non-decreasing in* W^Y *for each* X, H^X *such that* $(X, H^X, W^Y) \in \mathbb{S}^*_N$.
2. $J(X, H^X, W^Y)$ *satisfies Assumption 2 for* $\mathbb{S}^* = \mathbb{S}^*_N$.
3. $J(X, H^X, W^Y)$ *is jointly concave in* H^X, W^Y *for each* X *such that* $(X, H^X, W^Y) \in \mathbb{S}^*_N$.
4. $D_n(X, H^X, W^Y)$ *satisfies Assumption 1 for* $\mathbb{S}^* = \mathbb{S}^*_n$ *for every* $n = \overline{1, N}$.
5. $C_{n-1}(\Delta H^X, X)$ *satisfies Assumption 4 for every* $n = \overline{1, N}$.
6. X_n *and* Y_n *are defined by* (5) *and* (6) *correspondingly.*
7. μ_n *is diagonal for every* $n = \overline{1, N}$.

Then the value function satisfies the equation

$$V_{n-1}(X, H^X, W^Y) =$$
$$= \sup_{Z\in D_n} \min_{G_n \in \mathcal{G}_n(K_n,E_n)} \sum_{i=0}^{m(K_n)} p^i(G_n, E_n) V_n\Big((1+\mu_n\Delta t_n + \mathrm{diag}(G_{i,n})\sqrt{\Delta t_n})X, Z,$$
$$\big(W^Y - (Z-H^X)^T X - C_{n-1}(H_n^X - H^X, S_{n-1})\big)(1+r_n\Delta t_n)\Big),\ n = \overline{1,N}, \quad (14)$$

$$V_N(X, H^X, W^Y) = J(X, H^X, W^Y), \quad (15)$$

where $\mathrm{diag}(G_{i,n})$ *is a diagonal matrix with elements of vector* $G_{i,n}$ *on the main diagonal.*

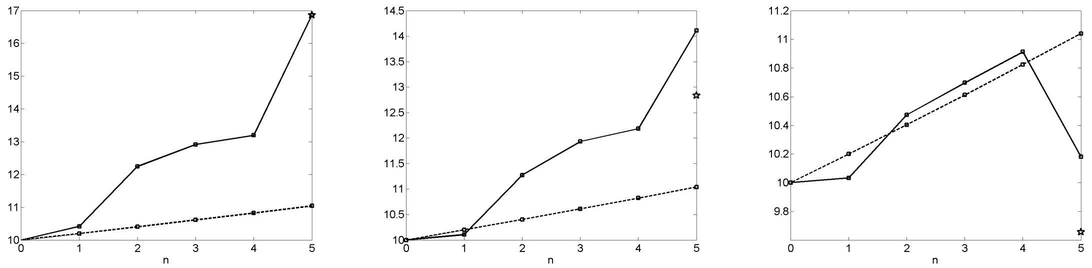
2 Numeric solution and computational results

The described method has been implemented for MatLab R2012a. We consider the case of isoelastic utility depending on the terminal liquidation value of the portfolio and trading limits of the form

$$D(X, H^X, W^Y) = \left\{ Z \in \mathbb{R}^m : \begin{array}{rcccl} -\beta^X W & \leq & Z^T X & \leq & (1+\beta^Y)W, \\ & & |Z|^T X & \leq & (1+\tilde{\beta}^Y)W \end{array} \right\}, \quad (16)$$

where $W = W^Y + {H^X}^T X$, so that D is compact and still satisfies Assumption 1 which can be readily verified. (16) can be interpreted as limits for total value of short positions including risk-free investments, and limits for the amounts invested in risky assets. Without the second constraint, the investment policy would allow to short some risky assets infinitely while investing in the other.

The value function can be calculated recursively according to (14)-(15), however this method becomes time consuming with the increase of the number of steps N and requires large amounts of memory to store the interim data which makes it impractical for the multi-step investment problem. Hence we use a step-by-step reconstruction of the value function on the state grid: first for t_N, then for t_{N-1} by using known values at t_N, and so on up to t_0. As a byproduct, we obtain reconstructed value function for the whole grid which can also be used for future analysis and strategy modeling if market parameters and forecasts are assumed unchanged.



(a) $\lambda = 0$ (b) $\lambda = 0.05$ (c) $\lambda = 0.12$

Figure 1: Portfolio market value W_n under the worst-case optimal strategy for various transaction costs coefficient values. Dashed line is the market value according to the risk-free strategy $H^X \equiv 0$. Pentagonal star denotes the liquidation value of the portfolio by the end of the strategy.

At time t_N value function is known from (15). To find V_{n-1}, V_n values are required in points both inside and outside the established grid. We interpolate and extrapolate V_n with the appropriate parametric approximation $\hat{V}_n$, which is concave for any values of the calibration coefficients. Then V_n fitting is reduced to simple linear regression problem. Since V_{n-1} depends solely on V_n, it is possible to calculate values of V_{n-1} on the grid in parallel mode.

To demonstrate the results, we consider one risky asset and stationary parameters $\mu_n \equiv \mu$, $E_n \equiv E$, $K_n \equiv K$. We assume that μ is given by an analyst with high forecast power to minimize influence from the forecasting errors. K is estimated and updated via a Bayesian method as the confidence interval for the detrended returns. E is set to zero to keep all the information about price expected behavior within μ. For the transaction costs function, we take a proportional model $C_{n-1}(\Delta H, X) = \lambda|\Delta H|X$, $n = \overline{1, N}$.

Figures 1a-1c present results for the worst-case strategy. Since the forecasting power is big enough, all decisions made by DSS were correct in terms of the long/short position, hence portfolio value has been increasing at every period. In the presence of costs, trade sizes have become smaller and the total profit has decreased. Further increase in λ shows that at some point costs are so large that risky investments are not practical, even if all the decisions are correct.

3 Conclusion

We apply the worst-case dynamic programming approach to the optimal portfolio selection problem in discrete time with terminal utility maximization criterion. We assume that the underlying model of the market is ambiguous but has some observable characteristics and is Markovian. One of the main assumptions is the compactness of the market parameters' range which can be considered a mild restriction if the range is chosen big enough. For the Markovian portfolio strategies we present the Bellman-Isaacs equation and provide sufficient conditions that allow to solve it numerically by simplifying the solution of the interim minimization problem over a set of probability measures. The conditions specify the required properties of the terminal utility function, the trading limit sets and the transaction costs function. All the properties are inherent to a wide range of practical cases and have economic interpretation.

While the worst-case approach can be used for guaranteed portfolio pricing and stress-testing on a short period, the current research is aimed at the strategical optimization, where the number of investment periods can be quite large. This raises the problem of the effective numeric solution that is practical in terms of calculation time and memory usage. We use the step-by-step reconstruction of the value function on a state space grid and use interpolation within the simple parametric class of functions to obtain values outside the grid.

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