Endogenous borrowing constraints and government debt in a life-cycle economy.

Preliminary and incomplete.

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Abstract

Consumption loans are not collateralized for young households. To prevent default on such loans, banks set borrowing limits, which may bind. I study a pure exchange, overlapping generations economy with limited commitment to loan repayment, so fully endogenous borrowing constraints. Defining a bubble as a useless asset with a positive market price, I allow financial intermediaries to create bubbles. In those economies where tight borrowing limits lead to permanent autarky, bubbles lead to positive borrowing limits and higher retirement savings, under favorable initial conditions. The result is due to the endogenous nature of borrowing limits. When agents expect high interest rates in the future, they rely on opportunities to save for retirement. This makes them avoid records on bankruptcy, so they are allowed to borrow more in equilibrium without risk of default.

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1 Introduction

Asset bubbles can have various nature and causes. Fully rational investors that can coordinate on one particular expected scenario of the economy's future can let the bubble exist only under particular market frictions, as shown by Santos and Woodford (1997). This paper discusses bubbles under two such empirically relevant frictions: non-Ricardian households and limited commitment to repay debts.

The latter friction is linked to the existence institute of personal bankruptcy in the United States. Namely, if an individual files personal bankruptcy under Chapter 7 of

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the US bankruptcy code, she is in many cases granted ,,a fresh start“: a protection of *labor* earnings from the creditors' garnishment. At the same time, the *financial* income of the bankrupt person, beyond for certain exemptions, must be used to compensate the harmed creditors. The credit that I focus on is consumption loans that have no underlying asset serving as collateral. Given possibility of personal bankruptcy, a person cannot fully commit to repay such a loan with future labor earnings. However, there is some degree of commitment, as garnishment of financial income after the bankruptcy creates an opportunity cost. Limited commitment to repay debts is a source of asset scarcity in an economy. As discussed in Caballero (2006), asset bubbles can potentially be a way for the financial market to absorb the excess supply of savings. An important goal of this paper is to show a case for rational bubbles to improve not only savers', but also borrowers' consumption smoothing — through the working of endogenous borrowing constraints.

The framework of the paper is a pure exchange, overlapping generations economy in which young borrowers cannot commit to repay consumption loans. Identical households live for three periods and receive perishable, age-specific endowments. These endowments are deterministic, exogenous, and are chosen so that the life-cycle income pattern is hump-shaped. As a result, households demand consumption credit when young and supply credit at pre-retirement age, in line with the life-cycle hypothesis of Modigliani and Brumberg (1954). Young agents have a possibility to default on their debts, and their subsequent endowments will not be seized. Financial intermediaries that take out all loans, can only seize retirement savings of defaulters. I call this state ,,financial autarky“ or simply ,,autarky“. By choosing to default, the households choose large consumption immediately after default and no consumption smoothing afterwards. The incentive to do so depends on the interest on retirement savings the market is expected to offer. Under some preference and endowment profiles, the economy cannot have equilibrium interest rates high enough to revert households from default. These economies have no dynamic equilibrium except for perpetual autarky.

This is where I find a scope for welfare-improving bubbles. Financial intermediaries can start to trade an asset with no fundamental value, such as government debt in absence of primary surpluses of the government budget. A part of intermediaries' asset demand, generated by saving needs of the middle-aged, can be satisfied with the bubbly asset. The new asset improves saving opportunities, as typically found for *dynamically inefficient* economies in the OLG literature. However, there is more: in the new equilibrium credit supply is partially crowded out, so interest rates are higher and households have more incentives to save for retirement. Intermediaries can safely lend more to the young without risk of default, as financial autarky has a higher opportunity cost. I find, therefore, that both savers and borrowers are better off in the presence of bubbles. I show an emergence of a steady-state as well as the transitory paths that are a Pareto improvement with respect to the autarky. This is achieved as bubbles improve the performance of the credit market: borrowing limits for young agents are strictly positive on the entire transitory path.

This paper builds on a model of Azariadis and Lambertini (2003) (AL henceforth).

The authors provide a full characterisation of dynamic equilibria in pure-exchange, overlapping generations environment with three periods of life and endogenous borrowing constraints. I adopt their framework, restrict parameter values somewhat, and allow bubble formation in the financial market. I leave out, at the same time, questions of indeterminacy of equilibria and full global dynamics analysis, that are important in the work of AL.

The welfare-improving properties of bubbles in the current work has the same nature as in the classical OLG literature: Samuelson (1958), Diamond (1965), Tirole (1985). The economies that I study have an interest rate strictly lower than population growth in an equilibrium without bubbles. This creates an opportunity to increase consumption of all generations without violation of the resource constraint. What is new in this paper is the additional intertemporal nature of this effect. It is the increase in *anticipated* interest rates that allows for more borrowing, not the increase in the current interest on savings.

A growing literature studies rational bubbles in exchange economies with limited commitment to repayment. In a seminal paper, Kocherlakota (2008) has introduced a concept of ,,injecting bubbles in the borrowing limits“. It has been shown that bubbles of different sizes can be added to equilibrium asset prices, if initial debts of borrowing constrained agents are lowered by a corresponding amount. Bubble crowds out credit in a one to one proportion.Hellwig and Lorenzoni (2009) have shown for a different punishment after default that all agents are able to borrow positive amounts in the equilibrium. A recent contribution of Bidian (2016) has extended the result to different punishment schemes: a full permanent exclusion from trade, a temporary exclusion from trade, and an exclusion from trade for a random number of periods, all of which are used in models of limited commitment. Summing up, the current consensus seems to be that credit under limited commitment and asset bubbles are *perfect substitutes*. This paper significantly departs from this point of view. The reason is the life-cycle problem of households, as opposed to infinite horizon smoothing across periods and states of the Bewley model. The bubble in the latter is traded by the constrained borrowers: they can buy it in the high-income period to sell on the ,,rainy day", where borrowing constraints bind. In this paper the constrained young agents do not participate in the trade of the bubble. All bequests are ruled out, so an agent has no balances of the bubbly asset in the beginning of life. Neither do they buy the asset by the end of the period, because they are net borrowers according to life-cycle hypothesis of Modigliani and Brumberg (1954). In equilibrium, as a result, an increase of bubble pulls up the interest rate, while the opposite holds in the Bewley-type models.These models, therefore, have bubbles deteriorate the welfare of savers. In the present model, consumption smoothing is improved both because of increased borrowing and more profitable saving.

To my knowledge, there is only one paper that addresses binding borrowing constraints of young consumersas a source of asset scarcity and bubbles: the work of Arce and Lopez-Salido (2011) on housing bubbles. The modelling strategy of the present paper is closely related to the one in this work. As myself, Arce and Lopez-Salido (2011) populate the economy with generations of agents living for three periods and receiving exogenous consumption good. Another good in their setting is housing, that is, as the endowment good, in an exogenous constant supply. Crucially, it is the value of housing that pins down the borrowing limits for young household. In this sense, borrowing limits are also endogenous in the model, but still hinge on an ad hoc parameter of constraint tightness, which is not the case in my model. The mechanism for bubble creation mirrors the one in the present model: the constraints for young borrowers are too tight, so the saving of the middle-aged is too big to be transformed into loans. An interesting implication of the authors' modelling is that bubbles on intrinsically useless assets and those on housing do equally well to alleviate asset scarcity. This is in line with the stance in Caballero (2006): some bubbles can be more dangerous than others, but an existence of *some* bubble is a necessity when economy lacks assets. The rest of the paper is organised as follows. In the next section I lay out the model of AL augmented with an additional ,,bubbly" asset. Original notation of the work is kept. The following section gives the results on the steady-state equilibria and local dynamics. Finally, some global dynamics properties of the model are presented with a help of simulated examples. The last section concludes.

2 Framework

2.1 Households.

In this section I reproduce the household problem from Azariadis and Lambertini (2003). The original notation is kept for all variables. The economy consists of overlapping generations of identical households, with $N_t = (1+n)^t$ being the number of households at period t. There is one good in the economy, and, importantly, it is non-storable. Each household gets exogenous real labour income in each period of life. The length of life is fixed at three periods, and the real labour income in the three periods is y_0, y_1 and y_2. The periods correspond to youth, middle age and retirement, so y_2 is to be interpreted as pension income that does not result from a households' financial investment. The problem of each individual is to smooth consumption between three periods of life in the presence of borrowing constraints that are discussed in later sections. An individual born in period $t \geq 1$ has a lifetime utility function of the following form:

$$V_t^t = u(c_t^t) + \beta u(c_{t+1}^t) + \beta^2 u(c_{t+2}^t)$$

where u is continuously differentiable and concave. The upper time index is the period of birth of the individual: c_t^v is the quantity consumed in period t by an individual born in v. We consider the consumption smoothing problem of an individual born in t who can save and borrow against future income, with interest rates and borrowing constraints given by the market:

$$\max_{c_t^t, c_{t+1}^t, c_{t+2}^t} \quad u(c_t^t) + \beta u(c_{t+1}^t) + \beta^2 u(c_{t+2}^t) \tag{1}$$

$$\text{s.t.} \begin{cases} c_t^t + \frac{c_{t+1}^t}{R_t} + \frac{c_{t+2}^t}{R_t R_{t+1}} & \leq y_0 + \frac{y_1}{R_t} + \frac{y_2}{R_t R_{t+1}}, \\ y_0 - c_t^t & \geq B_t^c, \\ y_1 + R_t(y_0 - c_t^t) - c_{t+1}^t & \geq \bar{B}_{t+t}^c, \end{cases} \tag{2}$$

where R_t is the gross interest on a unit of consumption good lent out at t to be repaid at $t+1$. B_t^c and $\bar{B}_{t+1}^c$ are the negatives of borrowing limits that an agent born in t faces in youth and in middle age, respectively. The borrowing conditions are discussed in section 2.4. Based on excess demands of agents on a given date, they can either borrow or lend the perishable consumption good. I label this positive or negative asset positions on the credit market. It is convenient to formulate the agent's problem in terms of end of period asset positions. A variable b_t^t denotes the real amount of assets at the end of t of an agent born in t, so:

$$b_t^t = y_0 - c_t^t \tag{3}$$

Accordingly, b_{t+1}^t is the position of the same agent by the end of $t+1$, which also depends on assets of previous period:

$$b_{t+1}^t = y_1 + b_t^t R_t - c_{t+1}^t \tag{4}$$

The amount of good that the middle-aged of period $t+1$ have to repay after borrowing in youth is denoted by a variable w_{t+1}:

$$w_{t+1} \equiv -R_t b_t^t \tag{5}$$

Debt repayment of the middle aged at the beginning of the period will be used as one of the state variables of the economy.

We can now rewrite the problem in terms of asset positions of the young and the middle-aged, as well as debt repayment of the middle-aged:

$$\max_{b_t^t, b_{t+1}^t, c_t^t, c_{t+1}^t, c_{t+2}^t, w_{t+1}} \quad u(c_t^t) + \beta u(c_{t+1}^t) + \beta^2 u(c_{t+2}^t) \tag{6}$$

$$\text{s.t.} \begin{cases} c_t^t + b_t^t & = y_0, \\ c_{t+1}^t + b_{t+1}^t & = y_1 - w_{t+1}, \\ c_{t+2}^t & = y_2 + R_{t+1} b_{t+1}^t, \\ w_{t+1} & = -R_t b_t^t \\ b_t^t & \geq B_t^c, \\ b_{t+1}^t & \geq \hat{B}_{t+1}^c; \end{cases} \tag{7}$$

In the first period some agents are already in retirement age, others in middle age. The former only consume their income:

$$c_1^{-1} = y_2 + (1+n) w_1, \tag{8}$$

where w_1 is an exogenous amount of good that each household born in 0 owes to that born in 1. One has to specify this value as an initial condition for the dynamic equilibrium of the economy.

The middle-aged of period 1, that were born in 0, face a truncated version of the problem (6)-(7):

$$\max_{b_1^0, c_1^0, c_2^0, w_1} \quad u(c_1^0) + \beta u(c_2^0) \tag{9}$$

$$\text{s.t.} \quad c_1^0 + b_1^0 = y_1 - w_1 \tag{10}$$

$$c_2^0 = y_2 + R_1 b_1^0 \tag{11}$$

2.2 Agents' Saving Functions and the Financial Sector

To simplify the agents' policy functions, I introduce a series of conventional assumptions for preferences and endowments.

Assumption 1. *Consumption in different periods of life is gross substitutes and normal goods.*

The assumption refers to both preferences and endowment profiles. For example, with CIES utility functions, the sufficient condition for the Assumption to hold is IES $\geq$ 1. With $0 <$ IES < 1, the sufficient condition becomes $y_1 > y_0 \geq y_2$.

Asset demand for slack borrowing constraints (7) is characterized in the following Proposition.

Proposition 1 (Azariadis and Lambertini (2003))**.** *Unconstrained demand for assets of an individual born in t, b_t^t and b_{t+1}^t, is the solution of problem* (6) *with respect to* (7)*. It is represented by continuously differentiable functions: $b_t^t = B(R_t, R_{t+1}^e)$ and $b_{t+1}^t = z(R_{t+1}, w_{t+1})$, such that:*

- $\frac{\partial B}{\partial R_t} > 0, \frac{\partial B}{\partial R_{t+1}^e} > 0$
- $\frac{\partial z}{\partial R_{t+1}} > 0, \frac{\partial z}{\partial w_{t+1}} < 0.$

Proof. From (3), (4), b_t^t and b_{t+1}^t are defined via the demand for consumption in t and in $t+1$. By gross substitutability, demand for current consumption is decreasing in current and future interest rates, hence $\frac{\partial B}{\partial R_t} > 0, \frac{\partial B}{\partial R_{t+1}} > 0$ and $\frac{\partial z}{\partial R_t} > 0$. Normality of old-age consumption implies a positive income effect on saving, so $\frac{\partial z}{\partial w_t} < 0$ as debt repayment reduces lifetime income. □

The deals on consumption loans between generations are mediated by perfectly competitive financial intermediaries. Namely, these intermediaries receive savings of one generation as deposits and use them to give out loans to another generation. Due to perfect competition, interest rate is the same for loans and deposits. In this sense, equilibrium outcomes are not affected by intermediaries: loans could be contracted by households directly. Intermediaries' actions matter when they start to create bubbles.

I assume an existence of an asset without fundamental value that is available for trade between the intermediaries. When the asset has market value, it is a pure bubble. As I will consider equilibria with rational expectations (perfect foresight), it is a rational bubble, as depicted in the title of the paper. A crucial structural assumption of the model is that the bubble asset cannot be traded by households and has no impact on them but via equilibrium interest rates. The asset is in fixed supply of D units and its period t price in consumption units is p_t^d. As deposits and loans are measured in consumption units, total bubbly asset position in period t, measured in consumption units, will be denoted D_t. Using the notation, market clearing condition for the bubbly asset can be stated as follows:

$$\frac{D_t}{p_t^d} = D \tag{12}$$

Bringing together the condition for periods t and $t+1$, we get:

$$\frac{D_t}{p_t^d} = \frac{D_{t+1}}{p_{t+1}^d} \tag{13}$$

$$\Leftrightarrow \frac{D_{t+1}}{D_t} = \frac{p_{t+1}^d}{p_t^d} \tag{14}$$

Equilibrium prices of the bubbly asset must exclude arbitrage between consumption loans and this asset:

$$\frac{p_{t+1}^e}{p_t} = R_t \tag{15}$$

2.3 Equilibrium with slack borrowing constraints.

The first step to discuss the equilibrium of the model is to assume that borrowing constraints $b_t^t \geq B_t^c$ and $b_{t+1}^t \geq \bar{B}_{t+1}^c$ are slack. This is always the case in an economy where agents can commit to repay their loans by pledging their future endowments. I will refer to this assumption as to full commitment.

For the goods market of period t to clear, total assets and liabilities of financial intermediaries must clear:

$$N_t b_t^t + N_{t-1} b_t^{t-1} = D_t \tag{16}$$

Recalling saving functions of agents, population growth at a constant rate n, and denoting d_t the total bubble in consumption units per middle-aged person of period t, the per capita expression is obtained:

$$(1+n)B(R_t, R_{t+1}^e) + z(R_t, w_t) = d_t \tag{17}$$

For the dynamic equilibrium to be expressed in per capita variables only, market clearing for the bubbly asset must be rewritten:

$$\begin{aligned} \frac{D_{t+1}}{D_t} &= \frac{p^d_{t+1}}{p^d_t} \\ \Leftrightarrow \frac{N_{t+1}d_{t+1}}{N_t d_t} &= \frac{p^d_{t+1}}{p^d_t} \\ \Leftrightarrow (1+n)\frac{d_{t+1}}{d_t} &= \frac{p^d_{t+1}}{p^d_t} \end{aligned}$$

Finally, using the no-arbitrage condition, one obtains

$$(1+n)\frac{d_{t+1}}{d_t} = R_t \tag{18}$$

Throughout the paper, perfect foresight is assumed to hold in equilibrium[1], so the expectations index is dropped for both interest rates and bubble prices:

$$R^e_{t+1} = R_{t+1} \tag{19}$$
$$p^e_{t+1} = p_{t+1} \tag{20}$$

The following equations and an initial condition for w_1 describe sequences $(R_t, w_t, d_t)_{t=1}^{\infty}$ which constitute a *dynamic equilibrium with perfect foresight and full commitment*:

$$(1+n)B(R_t, R_{t+1}) + z(R_t, w_t) = d_t, \tag{21}$$
$$(1+n)w_{t+1} = R_t(z(R_t, w_t) - d_t),\ t > 1 \tag{22}$$
$$(1+n)d_{t+1} = R_t d_t \tag{23}$$
$$-y_2 \leq w_t \leq y_1 \tag{24}$$
$$z(R_t, w_t) \geq -y_0 \tag{25}$$
$$0 \leq d_t < (2+n)y_0 + y_1 \tag{26}$$

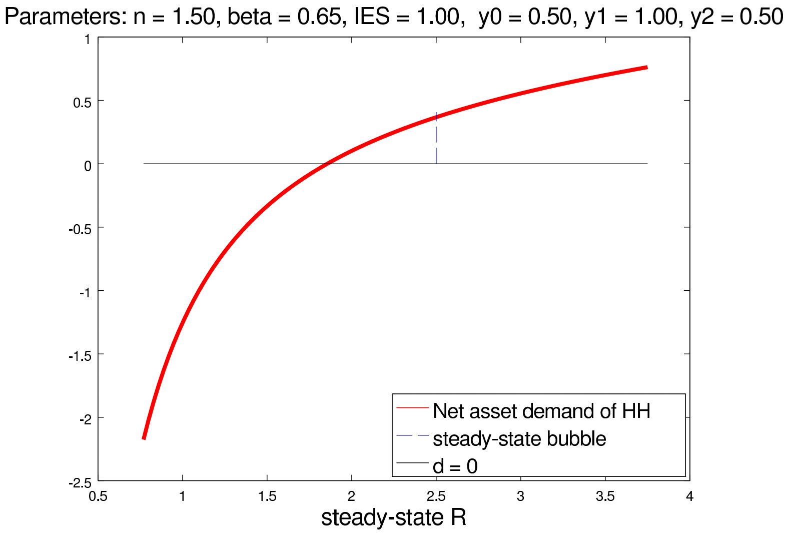
The last three inequalities restrict the state space of the system to exclude such states where a young, a middle-aged[2] or a retired household's intertemporal budget constraint is violated. In particular, the last equation restricts the bubble to have positive value bounded by the aggregate endowment available for investment.

Properties of these sequences can be deduced from the analysis of Kehoe et al. (1991). They are summarized in a proposition in the end of the section. To study steady-state equilibria, a steady-state net asset demand function should be analysed. Plugging the definition of w_t in the LHS of asset market clearing condition (21), one obtains:

$$(1+n)B(R_t, R_{t+1}) + z(R_t, R_{t-1}B(R_{t-1}, R_t))$$

[1]existence of perfect foresight under adaptive expectations in this model could potentially be treated with methods proposed in Bohm and Wenzelburger (2004)

[2]the repayment are assumed $< y_1$, so never to induce a middle-aged to borrow. This assumption is motivated by the endogenous borrowing constraints discussed below, that preclude middle-aged borrowing.



If the interest rates are all assumed to be constant, the following univariate function can be analysed:

$$B^u(R) = (1+n)B(R,R) + z(R, -RB(R,R)) \tag{27}$$

This is the net demand for assets of the economy as a function of a steady interest rate. The superscript u refers to "unconstrained", as the saving functions are only valid for slack borrowing constraints. This function is monotone increasing, which is a consequence of gross substitutability of consumption. Indeed, a rise in interest rates must decrease consumption in young age, increasing the first term. Furthermore, it must increase consumption in old age, increasing the second term, savings in middle age. By monotonicity of the net savings function, at most two steady states can exist. The first one features zero net private savings and no government borrowing. The second one, if exists, features positive net private savings. Figure 2.3 illustrates the case with two steady states.

Assumption 2 (MRS in autarky)**.** *Marginal rate of substitution in autarky is smaller than the rate of population growth for the young and greater for the middle-aged:*

$$\frac{u'(y_1)}{\beta u'(y_2)} < 1+n < \frac{u'(y_0)}{\beta u'(y_1)} \tag{28}$$

An autarky in the present model is a state where each individual at each age consumes the corresponding endowment. Given the monotonicity of demands for assets, Assumption 2 implies the young borrow and the middle-aged save if the gross interest rate is close to $1+n$.

Proposition 2 (Equilibria with no borrowing limits)**.** *If Assumptions 1 2 hold, equilibria of the economy with no borrowing limits have the following characteristics:*

- *At most two steady-states:*

1. *One that exists in any specification, with interest rate* $R^u \in \left[\frac{u'(y_1)}{\beta u'(y_2)}, \frac{u'(y_0)}{\beta u'(y_1)}\right]$ *and no bubble;*

2. *Another that exists in dynamically inefficient economies with* $R^u < 1 + n$*: a Golden Rule steady state with interest rate* $1+n$ *and positive per capita bubble* d^{ss}*.*

- *In case* $R^u < 1 + n$*, there is an infinity of dynamic equilibria with perfect foresight that converge to the steady-state without bubbles and exactly one dynamic equilibrium with perfect foresight that converges to the Golden Rule steady state. This holds for any initial condition* w_1*.*

- *In case* $1 + n < R^u$*, there is exactly one dynamic equilibrium converging to*

The proof can be found in Kehoe et al. (1991).

A Golden Rule steady state with $R = 1+n$ is named so according to the tradition in dynamic economics: this is the equilibrium that maximises consumption of the steady state representative household. This equilibrium is dynamically efficient, while the one without bubble can have a Pareto improvement if consumption of the initial retirees is increased. The set of equilibria in an economy with full commitment is quite standard. Qualitatively, the economy with three periods of life and gross substitutability is almost identical to the seminal model of Gale (1973) with two periods of life.

2.4 Limited Commitment and Endogenous Borrowing Constraints

Following AL, I assume limited commitment to loan repayment in the following sense. Agents can default at the beginning of their middle age on loans taken out in the young age. In this case, the financial intermediary has no possibility to seize the endowment income of the individual. Instead, all the subsequent income received by the defaulter from the financial market can be seized. Namely, if these agents are lenders in middle age, the intermediary claims the principal and interest that the agent is ought to receive in old age. This legal structure results in borrowing limits called *not too tight borrowing limits* in Alvarez and Jermann (2000), that depend on the future interest rates. With an additional assumption of perfect information of the financial intermediary about the agents' preferences and endowments, it is possible to specify equations for the borrowing limits B_t^c and $\bar{B}_t^c$.

Let us begin with the borrowing limits for the middle-aged. A middle-aged individual could be willing to borrow against retirement income[3]. Whether the individual has defaulted before or not, borrowing in the middle age is not feasible because of the limited commitment. When the borrower enters old age, the financial intermediary has no way to get the loan back. Savings in the middle age are thus constrained to be positive:

$$b_t^{t-1} \geq 0, \ \forall t \tag{29}$$

[3]this is impossible in the steady states with slack constraints of Proposition 2, but possible out of steady state.

Another borrowing constraint applies to the young borrowers. The constraint is deduced from the individual rationality condition that must hold for every individual:

$$\max_{c_{t+1}^t, c_{t+2}^t, b_{t+1}^t} \left(u(c_{t+1}^t) + \beta u(c_{t+2}^t) \right) \geq u(y_1) + \beta u(y_2) \tag{30}$$

$$\text{s.t.} \quad \begin{cases} c_{t+1}^t + b_{t+1}^t = y_1 - R_t b_t^t \\ c_{t+2}^t = y_2 + R_{t+1} b_{t+1}^t \end{cases} \tag{31}$$

The condition reads as follows. Whatever the amount of assets in period $t+1$ of an individual born in t, the utility that can be reached after debt repayment must be higher then the utility that can be reached in autarky — otherwise, the individual rationally defaults on the debt and reaches a utility level supperior to autarky utility. The left hand side of the inequality depends on the size of repayment after borrowing in youth. To see this, substitute consumption in middle and old age from the budget constraint in (30):

$$\max_{b_t^{t+1}} \left(u(y_1 + R_t b_t^t - b_{t+1}^t) + \beta u(y_2 + R_{t+1} b_{t+1}^t) \right) \geq u(y_1) + \beta u(y_2) \tag{32}$$

where b_t^t is a negative value if the individual was a borrower in t.

As in Alvarez and Jermann (2000), the intermediary gives to each borrower such a loan that she is just indifferent between defaulting and repaying the loan. This corresponds to the IRC (32) holding as equality. From the resulting equation one can derive the limit on debt repayment of the middle-aged agent, $R_t b_t^t$ (a negative value), as an implicit function of interest rate R_{t+1}: $R_t b_t = f(R_{t+1})$. The corresponding borrowing limit for the person in t , namely B_t^c from (2), is $f(R_{t+1})/R_t$. The properties of this important function are summarised in the following Proposition.[4]

Proposition 3 (Azariadis and Lambertini (2003))**.** *In the economy with limited commitment,*

$$b_t^t \geq f(R_{t+1})/R_t,$$

where the function $f(\cdot) : \mathbb{R}_+ \to \mathbb{R}_-$*:*

1. *is implicitly defined for* $R_{t+1} \in [u'(y_1)/\beta u'(y_2), \infty)$ *as*

$$\max_z \left(u(y_1 + f - z) + \beta u(y_2 + R_{t+1} z) \right) = u(y_1) + \beta u(y_2); \tag{33}$$

2. *is continuously differentiable and monotonically decreasing;*

3. $f'(R) = z(R, -f(R))/R$*;*

4. $f(u'(y_1)/\beta u'(y_2)) = 0$.

[4] Azariadis and Lambertini (2003) also provide an intuitive graphical explanation for the properties of $f(\cdot)$, see pp. 468-469.

The first three properties result from the implicit function derivation and the envelope theorem applied to (33). The last one comes from the fact that $z(u'(y_1)/\beta u'(y_2), w_t) \leq 0$ for any $w_t \geq 0$.

Before the introduction of an equilibrium in the limited commitment economy, another result of AL will be exposed. It allows to discriminate between slack and binding borrowing constraints for young agents in period t:

Proposition 4 (Azariadis and Lambertini (2003))**.** *The young are constrained in* t *if, and only if,* $R_{t+1} < m(R_t)$*, where:*

1. m *is a continously differentiable function* $[0, u'(y_0)/\beta u'(y_1)] \to \mathbb{R}_+$
2. $m' < 0$
3. $m(u'(y_0)/\beta u'(y_1)) = u'(y_1)/\beta u'(y_2)$
4. $R^* \in [u'(y_1)/\beta u'(y_2), u'(y_0)/\beta u'(y_1)]$ *is the unique fix point of* m*.*

   The middle-aged are borrowing constrained in t *if, and only if,* $R_t < u'(y_1)/\beta u'(y_2)$*.*

Proof. See Appendix 2 in Azariadis and Lambertini (2003). □

2.5 Equilibria with Endogenous Borrowing Constraints.

The previous section defines a rule of equilibrium borrowing limit determination, $B_t^c = f(R_{t+1})/R_t$. Bringing back the equilibrium perspective, recall that both loans and the bubble are assets for intermediaries, and the demand for both is determined by the current and future interest rates. One can think of intermediaries' operation in the economy as follows. Taking current interest rate, and the future one – due to perfect foresight – as given, the intermediary sets borrowing limits equalling $-f(R_{t+1})/R_t$. In case $R_{t+1} < m(R_t)$, when these limits bind, the young receive a loan equal to $-f(R_{t+1})/R_t$. Otherwise, they receive all the funds they demand, $-B(R_t, R_{t+1})$ (defined in Proposition 2.2). I label these situations constrained and unconstrained regime, respectively. Lastly, if current and anticipated interest rates are such that a bubble is traded, the intermediary adjusts its bubbly asset position based on current price p_t and anticipated p_{t+1}, that satisfy $p_{t+1}/p_t = R_t$.

The bond market equilibirum, or the assets–liabilities balance of the financial intermediary, is now given by

$$(1+n)\max\{f(R_{t+1})/R_t, B(R_t, R_{t+1})\} + \max\{0, z(R_t, w_t)\} = d_t, \tag{34}$$

This condition is equilvalent to (21) if the economy is in unconstrained regime in t. The critical difference of a limited commitment economy is that borrowing of the young is no longer monotone in the next-period interest rate R_{t+1}. As long as R_{t+1} is larger than $m(R_t)$, borrowing decreases in R_{t+1} by gross substitutability. When $R_{t+1} < m(R_t)$, however, borrowing increases in R_{t+1} as the constraint is relaxed in anticipation of larger saving in $t+1$. Temporary equilibria in both constrained and unconstrained regime are generally possible along a dynamic equilibrium with perfect foresight. At the same time, borrowing constraints make some states of the full commitment economy's state space impossible. The reason is as follows. GIven R_t, w_t, d_t, the aggregate demand for assets may be too high, with respect to the size of the bubble, for any anticipated R_{t+1}. The reason is exactly the non-monotonicity of the young agents' borrowing in the expected interest rates. The condition excluding these states from the state space is as follows:

$$(1+n)B(R_t, m(R_t)) + z(R_t, w_t) \le d_t \tag{35}$$

Perfect foresight equilibria of an economy under limited commitment are given by the following system:

$$(1+n)\max\{f(R_{t+1})/R_t, B(R_t, R_{t+1})\} + \max\{0, z(R_t, w_t)\} = d_t, \tag{36}$$
$$(1+n)w_{t+1} = \max\{-f(R_{t+1}), R_t(z(R_t, w_t) - d_t)\} \tag{37}$$
$$(1+n)d_{t+1} = R_t d_t \tag{38}$$
$$(1+n)B(R_t, m(R_t)) + z(R_t, w_t) \le d_t \tag{39}$$
$$w_1 \text{ given} \tag{40}$$
$$-y_2 \le w_t \le y_1 \tag{41}$$
$$0 \le d_t \le (2+n)y_0 + y_1 \tag{42}$$

If the whole dynamic equilibrium of the economy is in unconstrained regime, the system defining equilibria boils down to the one of the full commitment economy, with an addition of equation (40). The converse case,

3 Steady states: properties and local dynamics

To study the steady-states of the economy, one has to consider a function of steady-state net saving, analogous to (27):

$$\bar{B}(R) = (1+n)\max\{B(R,R), f(R)/R\} + \max\{0, z(R, \max[-RB(R,R), -f(R)])\} \tag{43}$$

The following result of AL is central in the subsequent analysis.

Proposition 5 (Steady State Regime)**.** *A steady-state is in the constrained regime if, and only if, steady state interest rate is below R^*.*

Proof. By definition of a fix point, $m(R^*) = R^*$. As $m' < 0$, $R < R^* \Leftrightarrow m(R) > R$. The economy with a stady state interest rate R is in this case in a constrained regime by Proposition 4. □

According to the Proposition, for $R > R^*$ the net savings function $\bar{B}(R)$ is equivalent to the net savings function in the full commitment case, $B^u(R)$, defined in (27). As proven before, that function is monotonously increasing. In order to fully characterize the properties of $\bar{B}$ for $R < R^*$, a technical assumption is made:

Assumption 3 (Azariadis and Lambertini (2003)). *For all* $R \in (u'(y_1)/\beta u'(y_2), R^*)$ *such that* $\bar{B}(R) = 0$*, the derivative* $\bar{B}'$ *is negative. A sufficient condition is:*

$$\forall (R, w) \in \mathbb{R}^2_+ : \frac{R}{z(R,w)} \frac{\partial z(R,w)}{\partial R} + \frac{\partial z(R,w)}{\partial w} < n.$$

The Assumption, used by AL, guarantees that $\bar{B}$ has a positive slope for any $R < R^*$ such that $\bar{B}(R) = 0$. In other words, the function always intersects the zero line from below for $R < R^*$. This allows to give a full characterization of the shape of the function. It will depend on the specification of the economy, namely, the instanteneous utility function $u(\cdot)$, the discount factor β, the endowment profile (y_0, y_1, y_2) and the population grwoth rate n. To infer the steady states of an economy, one needs to map the specification to two statistics. The first one is the interest rate in the steady-state with zero government debt under the assumption of full commitment — the R^u of Proposition 2. Secondly, one has to calculate R^*, the fix point of $m(\cdot)$. By ordering the three values one is able to infer the shape of the $\bar{B}$ function and so the number of steady states. The largest difference is between those economies that have $R^u < R^*$ and $R^u > R^*$.

The following Lemma summarizes results of AL on the shape of $\bar{B}(\cdot)$. It is used to determine the set of stationary states in the following sections.

Lemma 1 (Azariadis and Lambertini (2003)). *The following is true for* $\bar{B}(\cdot)$*:*

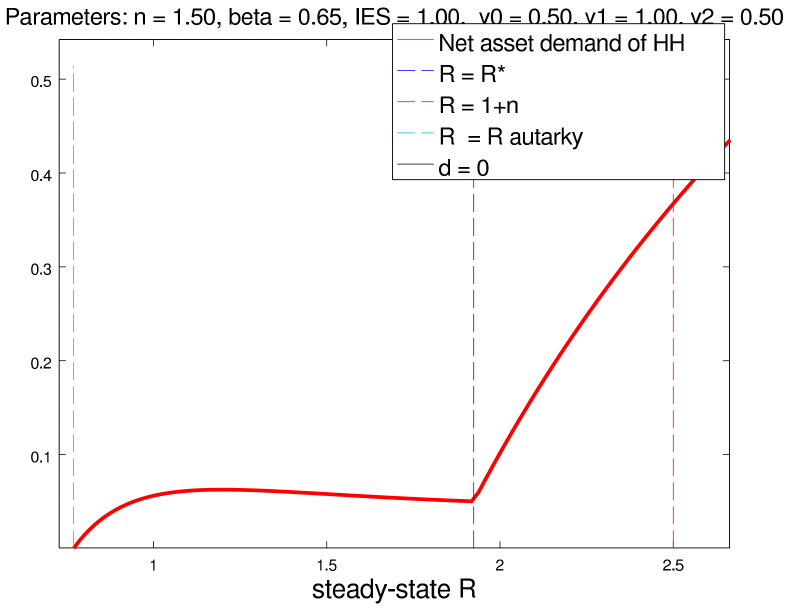
1. $\bar{B}(R) = 0$ *for all* $R \leq u'(y_1)/\beta u'(y_2)$
2. *it is positive for all* $R > u'(y_1)/\beta u'(y_2)$ *if* $R^u < R^*$
3. *has exactly two zeroes for* $R > u'(y_1)/\beta u'(y_2)$ *if* $R^u \geq R^*$

Proof. See Appendix A. □

In this paper I confine myself to economies that feature $R^u < R^*$, so that aggregate net demand for assets is always non-negative. Figure 3 shows a typical example of a graph of $\bar{B}(\cdot)$ in this case.[5] It is straightforward that all such economies have exactly two steady states. There are, however, two cases with qualitatively different dynamics and welfare properties. The first case, with high enough population growth, so that $1 + n > R^*$ is depicted in figure **??**. The two steady states are autarky, with $R = \bar{R}_1, d = 0$, and the Golden Rule state with $R = 1 + n, d > 0$ and unconstrained regime, according to Proposition 5. Thus the economy could be in a Golden Rule state instead of perpetual autarky in this case. However, to anticipate the dynamics discussion in **??**, convergence to this steady state is often impossible.

The second case, with $1 + n < R^*$, is depicted in figure **??**. Autarky is still one of the steady states, and the other state has $d > 0$ and is in constrained regime. In the two cases, welfare ranking of the steady states is straightforward: welfare is the lowest

[5] All the simulated examples throughout the paper are obtained with a felicity function $u(\cdot) = ln(\cdot)$ and parameter values depicted in the figures.



in autarky, first-best in the Golden Rule state and intermediate in the constrained state with positive bubble. Moreover, if the economy is not in autarky, it means some of the funds that intermediaries obtain from middle-aged are lent to the young, so bubble is never the only asset of the intermediary. This is because borrowing limits of the young are always stricly positive unless the economy is in autarky:

Proposition 6. *Suppose the specification of the economy is such that $R^u < 1+n < R^*$. Borrowing limit for the young agents, $f(R)/R$, is strictly positive in the steady state with a strictly positive stock of government debt.*

Proof. By Assumption 2, $1 + n > u'(y_1)/\beta u'(y_2)$, so the steady state interest rate is bigger than the MRS of the middle-aged agents. As $f(\cdot)$ is decreasing, $f(1 + n) < f(u'(y_1)/\beta u'(y_2)) = 0$, so the borrowing limit $\frac{-f(1+n)}{1+n}$ is strictly positive. □

The clear desirability of the non-autarkic steady states begs for analysis of their stability. In the following subsection, local dynamics around the steady states is explored. After that, I report the tentative analysis of the economy's global dynamics by means of a simulated example.

3.1 Local dynamics

3.1.1 Autarky

The economy in the constrained regime in autarky and in states that are close to it. This simplifies the analysis of dynamics, as equations (37) – (38) are replaced with one

$$(1 + n)\frac{f(R_{t+1})}{R_t} + z(R_t, -f(R_t)) = d_t \tag{44}$$

and the dynamics of w is given by the initial condition and $R_t = -f(w_t)$. We end up with a system with two dynamic variables, R_t and d_t. Any equilibrium trajectory is described by the following system:

$$(1+n)\frac{f(R_{t+1})}{R_t} + z(R_t, -f(R_t)) = d_t \tag{45}$$

$$(1+n)d_{t+1} = R_t d_t \tag{46}$$

This system is easily analysed by means of linearisation:

$$J(R,d) = \begin{pmatrix} -\frac{f(R)}{z(r,-f(R))} + \frac{\partial z}{\partial R}\frac{R^2}{(1+n)z(R,-f(R))} + \frac{\partial z}{\partial w} & -\frac{R^2}{(1+n)z(R,-f(R))} \\ \frac{d}{1+n} & \frac{R}{1+n} \end{pmatrix} \tag{47}$$

is the Jacobian of the system linearised around a steady state with interest rate and bubble R, d. In case of autarky, it becomes

$$\lim_{R\to\bar{R}_1; d\to 0} J = \begin{pmatrix} -\infty & -\infty \\ 0 & \frac{\bar{R}_1}{1+n} \end{pmatrix} \tag{48}$$

which has $\lim_{R\to\bar{R}_1}\lambda_1 = -\infty$; and $\lim_{R\to\bar{R}_1}\lambda_2 = \frac{\bar{R}_1}{1+n} < 1$ (by Assumption 2). It follows that autarky is a saddle in the two-dimensional dynamics of constrained regime. In other words, there is exactly one equilibrium converging to autarky. Note that this result does not depend on the parametrisation of the economy as long as Assumption 2 holds.

3.1.2 Constrained non-autarkic steady state

Local dynamics around steady state is only partially characterised.

$$J(1+n, d^c) = \begin{pmatrix} -\frac{f(1+n)}{z(1+n,-f(1+n))} + \frac{\partial z}{\partial R}\frac{1+n}{z(1+n,-f(1+n))} + \frac{\partial z}{\partial w} & -\frac{1+n}{z(1+n,-f(1+n))} \\ \frac{d^c}{1+n} & 1 \end{pmatrix} \tag{49}$$

To characterise the stability of the eigenvalues of this matrix, consider the trace and determinant of the matrix:

$$trJ = -\frac{f(1+n)}{z(1+n,-f(1+n))} + \frac{\partial z}{\partial R}\frac{1+n}{z(1+n,-f(1+n))} + \frac{\partial z}{\partial w} + 1 \tag{50}$$

$$\begin{aligned} detJ(1+n, d^c) &= -\frac{f(1+n)}{z(1+n,-f(1+n))} + \frac{\partial z}{\partial R}\frac{1+n}{z(1+n,-f(1+n))} + \frac{\partial z}{\partial w} + \frac{d}{z(1+n,-f(1+n))} \\ &= 1 + \frac{\partial z}{\partial R}\frac{1+n}{z(1+n,-f(1+n))} + \frac{\partial z}{\partial w} \end{aligned}$$

We first observe that the eigenvalues are necessarily complex:

$$1 + trJ + detJ > 0$$
$$1 - trJ + detJ > 0$$

It means that there is exactly one equilibrium that either converges or diverges from the steady state and has a spiral shape. The stability of the steady state is characterised only for a subclass of preferences: the CIES preferences with IES above unity.

Proposition 7 (Stability of the non-autarkic constrained steady-state)**.** *For a felicity function* $u(c) = \frac{c^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}}$ *with* $\sigma \geq 1$ *the non-autarkic steady-state is an unstable focus.*

Proof. Given that the eigenvalues of the Jacobian $J(1+n, d^c)$, given in equation (49), are complex conjugate, they are larger than 1 in modulus if and only if $det(J(1+n, d^c)) > 1$.

$$det(J(1+n, d^c) = 1 + \frac{\partial z}{\partial R}\frac{1+n}{z(1+n, -f(1+n))} + \frac{\partial z}{\partial w}$$

and $det(J(1+n, d^c) - 1$ has a sign of

$$(1+n)(-(\sigma-1)\beta^\sigma(1+n)^\sigma(1-f(1+n)) - y_2/y_1$$
$$-(y_2/y_1)\sigma\beta^\sigma(1+n)^(\sigma-1)) - \beta^\sigma(1+n)^\sigma((1-f(1+n))\beta^\sigma(1+n)^\sigma + (y_2/y_1)) < 0$$

□

3.1.3 Golden Rule steady state

Dynamics of the economy around the Golden Rule steady state are given by the same system as in the full commitment case. Therefore, it follows from Proposition 2 that it is a saddle in the three-dimensional system. For each initial w_1, there is exactly one pair R_1, d_1 that leads to convergence to the Golden Rule steady state.

4 Global dynamics: preliminary analysis

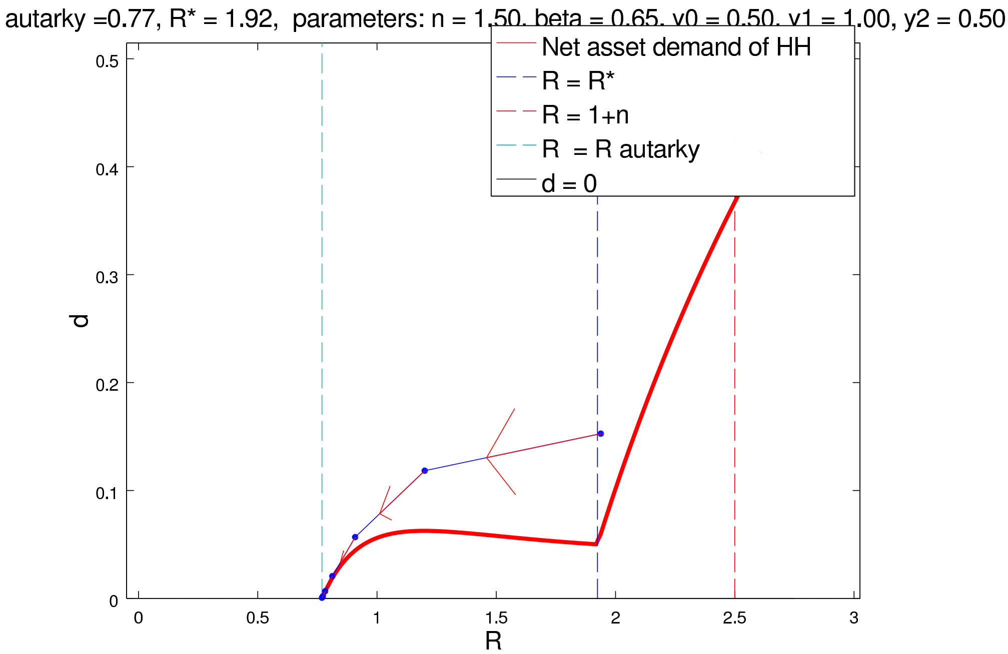
In this section first steps of global dynamic analysis are undertaken. In particular, I look at equilibrium trajectories converging to either one of two possible steady states. Two cases should be divided again: one where a non-autarkic steady state is a Golden Rule steady state and one where it is in constrained regime.

To avoid treatment of indeterminacy, I use the following trick for the analysis. Let us look at equilibria that have constrained young agents in the period preceding the first one, $t = 1$.[6] We are given the initial debt repayment of these households, w_1, as the initial condition. Using $w_t = -f(R_t)$ that holds in constrained regime, R_1 is obtained from the initail condition as

$$R_1 = f^{-1}(-w_1).$$

In the following analysis I can therefore treat R_1 as a predetermined variable and d_1 chosen by the agents so that the economy is on one of the dynamic equilibria with perfect foresight.

[6]It is empirically plausible to focus on thoose equilibria where young households always face binding constraints



4.1 Case $1 + n > R^*$

There are two steady-state equilibria are: autarky and a Golden Rule steady state. Figures 4.1, 4.1 show a projection of the three-dimentional state space on a two-dimensional manifold $w = -f(R)$.

The first observation is that convergence to autarky is a feasible equilibrium whenever borrowing constraints bind in $t = 0$, so R_1 is predetermined. As autarky is a saddle steady state of the two-dimensional system, there is exactly one initial bubble value b_1 that allows for convergence to autarky.

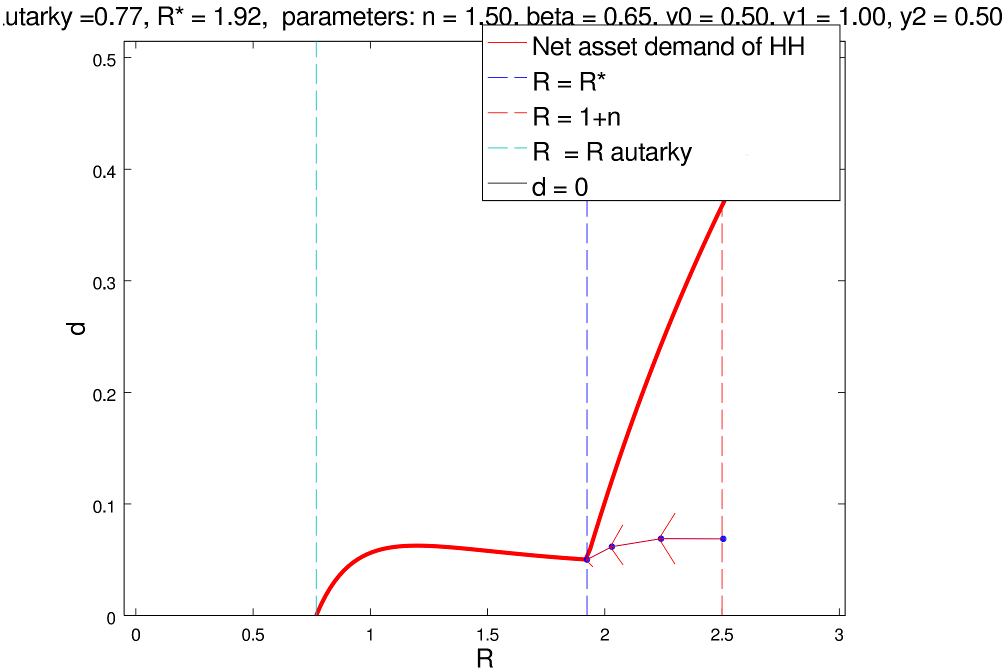
Secondly, for an equilibrium converging to Golden Rule steady state to be feasible, $R_1 \geq R^*$ is necessay, but not sufficient. Indeed, any equilibrium with constraints binding initially and slack in the long run must contain $(R^*, w^* = -f(R^*), d^* = (1+n)f(R^*)/R^* + z(R^*, w^*))$. Figure 4.1 showing a path in constrained regime encompassing (R^*, w^*, d^*), suggests this point is only attainable from a part of the state space to the right of R^*.

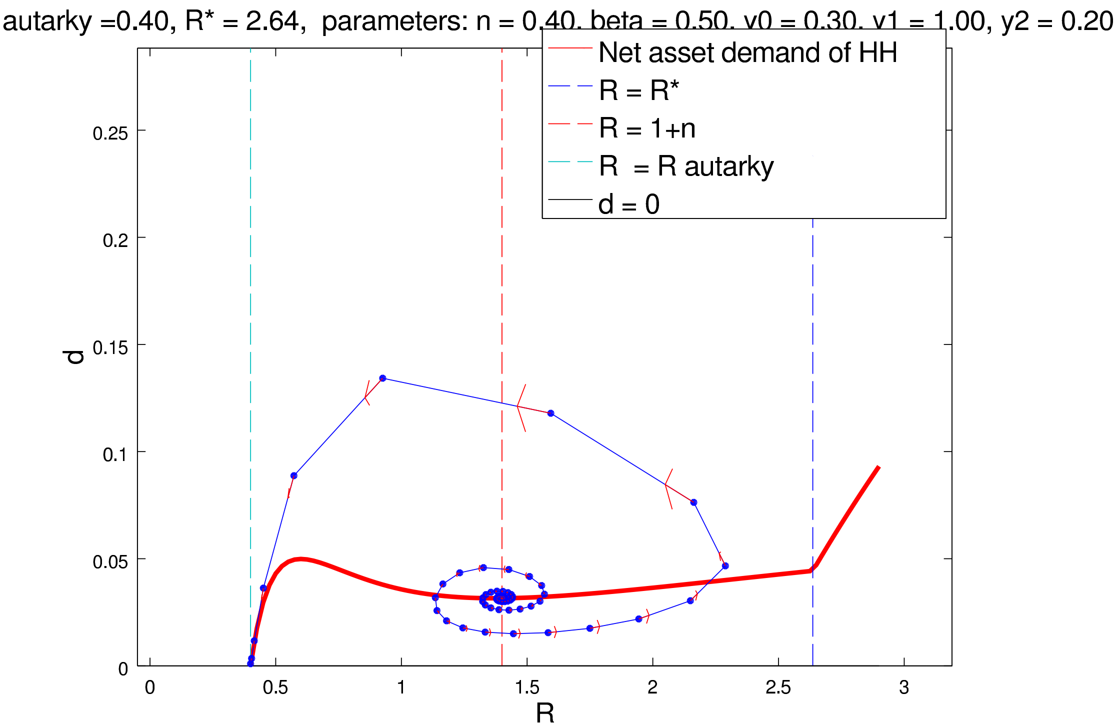
4.2 Case $1 + n < R^*$

In this case, there is a steady state with binding constraints and positive per capita value of bubble, instead of the Golden Rule steady state. It is presented in Figure 4.2.

Equilibria converging to the autarky have different properties from the case of the previous section. Namely, there are two new features:

1. For the predetermined R_1 in an interval $[R_1^{low}, R_1^{high}]$, this equilibrium is non-monotonic, with growing oscillations around the bubbly steady state.


.utarky =0.77, R* = 1.92, parameters: n = 1.50, beta = 0.65, y0 = 0.50, y1 = 1.00, y2 = 0.50
Net asset demand of HH
R = R*
R = 1+n
R = R autarky
d = 0
d
R


autarky =0.40, R* = 2.64, parameters: n = 0.40, beta = 0.50, y0 = 0.30, y1 = 1.00, y2 = 0.20
Net asset demand of HH
R = R*
R = 1+n
R = R autarky
d = 0
d
R

2. For $R_1 > R_1^{high}$, the equilibrium is infeasible.

The second finding can be understood intuitively. When the initial interest rate is fixed to be too high, initial saving is high, so one of the following events must realise in equilibrium. Firstly, initial borrowing constraints could be loose enough to absorb a considerable part of savings and prevent unsustainable bubbles. This requires R_2 to be quite high, making the situation reproduce in $t = 2$. Secondly, the saving could be absorbed by a large value of bubble, that requires even larger bubbles in the future. In a seminal paper on bubbles in exchange economies, Santos and Woodford (1997) find that bubble can be sustained in equilibrium only if the present value of aggregate endowment is infinite. In this economy, a predetermined R_1 that is too high causes $(R)_{t=2}^{\infty}$ to be high enough for the aggregate endowment to have finite value.

The preliminary analysis suggests the case with $R^* < 1+n$ is not always dominating the other one, despite the potential ability to converge to a first best steady state. Firstly, the existence of trajectories as in Figure 4.1 is not guaranteed. Secondly, it requires $w_1 > -f(R^*)$ as an initial condition, which is a relatively small set in the analysed example. Finally, the transition to autarky in case of $1 + n < R^*$ takes much longer. If the economy starts not too far from the unstable non-autarkic steady-state, the transition may take over 50 periods. The major hazard for this economy is to have w_1 and so R_1 too large, which leaves no possibility for a dynamic equilibrium with trade.

5 Conclusion

This paper has addressed a question of welfare effects of bubbles of financial market in a life-cycle economy with asset shortage grounded in limited commitment of the young to repay assets. In particular, I study economies from Azariadis and Lambertini (2003), where autarky is the only dynamic equilibrium with perfect foresight it absence of bubbles. When bubbles are introduced, equilibria with strictly positive borrowing limits, converging to autarky, emerge. The analysis of global dynamics at present only suggests that high indebtedness of the initial generation might prevent the existence of these new, favourable equilibria.

This paper has an important testable implication. If we think empirical economies correspond to equilibria of the model where borrowing constraints of the young are binding, borrowing of the young should be positively related to value of bubbles traded by the financial sector. A crucial structural restriction of the model is that value of bubbles must not have considerable direct influence on labour and investment income of the household sector. Of course, financiers that engage in the bubble creation are, after all, also a part of the economy's households, but model predictions hold only if this part is small.

Appendices

A Proof of Lemma 1

1. From Proposition 4, $R^* \geq u'(y_1)/\beta u'(y_2)$, so the young are constrained for R from the interval under consideration and $\bar{B}(R)$ becomes:
$$\bar{B}(R) = (1+n)f(R)/R + z(R, -f(R)), \;\; R \leq R^*. \tag{51}$$
Furthermore, $f(R)/R = 0$ and $z(R, -f(R)) \leq 0$. We conclude that $\bar{B}(R) = 0$ for all $R \leq u'(y_1)/\beta u'(y_2)$.

2. For $R \in (u'(y_1)/\beta u'(y_2), R^*]$, $\bar{B}(R)$ is positive by continuity and Assumption 3. $\lim_{R \searrow u'(y_1)/\beta u'(y_2)} \bar{B}'(R) \geq 0$, as $f'(R) = z(R, -f(R))/R \geq 0$ by Proposition 3 and $\partial z(R, -f(R))/\partial R > 0$ by gross substitutability. Furthermore, $\bar{B}(R^*) > 0$, as $(1+n)B(R^*, R^*) + z(R^*, -R^* B(R^*, R^*)) > 0$ and $(1+n)f(R)/R \geq B(R^u, R^u)$. Therefore, $\bar{B}(R)$ is strictly positive on $R \in (u'(y_1)/\beta u'(y_2), R^*]$, as by Assumption 3 $\bar{B}(R)$ can become zero only once on this interval.

   For $R > R^*$, $\bar{B}(R)$ is equivalent to the unconstrained net savings function:
$$\bar{B}(R) = (1+n)B(R, R) + z(R, -RB(R, R)), \;\; R > R^*. \tag{52}$$
This function is increasing by gross substitutability, so $\bar{B}(R) > 0$ for $R \geq R^*$ by continuity.

3. By monotonicity of $\bar{B}(R)$ for $R \geq R^*$, $\bar{B}(R^*) < 0$ as $R^* < R^u$ and $\bar{B}(R^u) = 0$. As $\lim_{R \searrow u'(y_1)/\beta u'(y_2)} \bar{B}'(R) \geq 0$, the function must become zero at least once on $R \in (u'(y_1)/\beta u'(y_2), R^*)$ by continuity. Given Assumption 3, it happens exactly once. The function, therefore, becomes zero once for $R \in (u'(y_1)/\beta u'(y_2), R^*)$ and once for $R \geq R^*$.

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