Types of nodes and centrality measures in networks

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Abstract

Equilibrium behaviors in games on network are often defined by centralities of players. We show that several centrality measures (degree; eigenvalue centrality; Katz-Bonacich centrality; diffusion centrality; two cases of alpha centrality) do characterize not just separate nodes but types of nodes. Our network typology uses the fact that the nodes in an undirected graph may be colored in a minimal number of colors in such way that any node of a color has definite numbers of neighbors of definite colors. Networks belonging the same typology have the same 'type adjacency matrix' T, which shows for each type numbers of neighbors of different types. For any typology, if i an j are nodes of the same type (may be even belonging different networks of this typology), then c(i)=c(j), where c is any of the above-mentioned centrality measures. Networks of different size but with the same typology have common properties in economic models; in particular, game equilibria may be transplanted among networks of the same typology. For calculation of any of the listed centrality measures, the matrix T may be used instead of the adjacency matrix. Bloch et al., 2017 formulate a problem: for which classes of networks each of a set of several centrality measures defines the same order on the set of nodes of network? They find such class of trees – so called, regular monotonous hierarchies. We show that such kind classes are not limited by trees: any typology with two types of nodes has this property for the above-mentioned set of centrality measures. The intersection of the classes of networks described in Bloch et al., 2017 and in our paper is the class of star networks.

1. Introduction

Models of network economics and network games are an object of a fast-developing body of literature. These models demonstrate that, because of network externalities, agents' behavior in equilibrium is defined by their positions in network, which are described by one or other centrality measure. However, different centrality measures prove important in different models: in particular, degree centrality – in a model of systematic shift of social norms (Jackson, 2017), betweenness centrality – in a Florentine families marriage case (Padgett, Ansell, 1993), eigenvalue centrality – in a model of aggregation of information by society (Golub, Jackson, 2010), Katz-Bonacich centrality – in a model of criminal behavior (Ballester et al., 2006), diffusion centrality – in models of diffusion in networks (Banerjee et al., 2013), a special case of alpha centrality (hereinafter referred to as alpha-gamma centrality) – in a model of knowledge production (Matveenko and Korolev, 2016, 2017). Such variety of centrality measures found to be of importance in various situations leads naturally to challenging questions about the nature of the centrality measures themselves. There is an emergent literature on the interrelations of different centrality measures and their relation with other structural characteristics of networks (e.g. König et al., 2014, Csató, 2017, Dequiedt, Zenou, 2017, Qiao et al., 2017, Bloch et al., 2017, Jackson, 2018).

In the present paper, we show that there is a set of centrality measures which characterize not just separate nodes of networks, but types of nodes selected by a rather universal structural characteristic – a network typology. The concept of network typology is based on the fact that in any network (undirected graph) $\langle M, N \rangle$ with a set of nodes $M = \{1,...,n\}$ and a set of edges *N*, the nodes may be colored in a minimal number of colors *S* in such way that any node of color *j* (*j*=1,…,S) has definite numbers of neighbors of each of S colors. Such coloring provides a division of the set of nodes *N* into *S* types. Correspondingly, to each network a 'type adjacency matrix' corresponds: $T = (t_{ij})$ of size $S \times S$, where t_{ij} is the number of neighbors of type *j* for any node of type *i*[1]. Two networks are said to have the same typology if they have the same type adjacency matrix.

A class of one-type networks is the familiar class of regular networks (such that all nodes in network have the same degree). The next in the order of increasing complexity is the class of networks with 2 types of nodes. It is a subclass of the class of networks with two degrees, such

[1] Our type adjacency matrix T is known in spectral graph theory as quotient matrix (e.g. Brouwer, Haemers, 2012, Atik, Panigrahi, 2015).

that each node of degree d_i $(i=1,2)$ has t_{i1} neighbors of degree d_1 and t_{i2} neighbors of degree 2 ($t_{i1}+t_{i2}=d_i$).

Each tree possesses a unique typology, while for each connected network which is not a tree (has a cycle) there exists a sequence of networks of the same typology with increasing network size.

The concept of network typology differs of other approaches to the structure of networks. We will see, for instance, that networks of similar size and similar typology may have different topological structure. In the same time, networks of similar size and similar distribution of degrees may have different typologies.

We prove that centrality measures of a class (degrees, eigenvector centrality, Katz-Bonacich centrality, diffusion centrality, and two cases of alpha centrality: alpha-beta centrality and alpha-gamma centrality) are defined by the network typology, i.e. depend only on types of nodes given typology. This implies that networks of different size and topology but with the same typology have common properties in many economic and game models. In particular, this implies that in many cases game equilibria may be transplanted among networks of the same typology.

We show that for calculation of the above-mentioned centrality measures, only the knowledge of typology is needed, but knowledge of the full structure of the network is excessive and is not necessary. Thus, the information containing in the type adjacency matrix T is enough for calculating the centrality measures of the above-mentioned class. For example, we show that Bonacich centralities in nodes of different types are given by the vector $(\tilde{I}-\alpha T)^{-1}\tilde{1}$, where T is the type adjacency matrix, $\tilde{I}$ is the $S\times S$ identity matrix and $\tilde{1}$ is the S-vector of all ones. Notice, that the size of the type adjacency matrix T may be several orders lower than the size of the original adjacency matrix of the network.

Opposite to some other centrality measures, the Katz-Bonacich centrality and the alpha centralities exist only in some regions of parameter α; we discuss the conditions of existence of these centrality measures. We show that possibilities to subjectively choose parameter α are rather limited because of the presence of network limitations on α. For example, for the Katz-Bonacich centrality to be defined in star network, α has to decline unlimitedly when the network grows. The Bonacich centrality and the alpha-gamma centrality never coexist under a joint α.

König et al., 2014 and Bloch et al., 2017 formulate the following problem: for which classes of networks each centrality measure of a set of centrality measures defines the same order on the set of nodes of network? In fact, two questions are there: about the class of networks and about the set of centrality measures. Bloch et al., 2017 introduce a class of trees – so called regular monotonous hierarchies – and prove (Proposition 2 and Corollary 1 in their paper) that, for any tree of this class, orders defined on the set of nodes by the following set of centrality measures: degree centrality, decay centrality, Katz-Bonacich centrality, diffusion centrality, intermediary centrality – do coincide. For each tree not belonging the class, there exists a pair of centralities from listed above which defines different orders on the set of nodes.

We show that such kind of classes of networks are not limited by trees. We prove that for the class of networks with two types of nodes, the orders defined on the set of nodes by the centrality measures of the above-mentioned class (degree, eigenvalue centrality, Katz-Bonacich centrality, diffusion centrality, alpha-beta centrality, alpha-gamma centrality) do coincide. The intersection of the classes of networks described in Bloch et al., 2017 and in our paper is the class of star networks.

The rest of the paper is organized as follows. In Section 2 we remind definitions of centrality measures used in the paper. In Section 3 we introduce the concepts of type of node and network typology. In Section 4 we establish a relation between the network typology and several centrality measures. In Section 5 we prove some mathematical properties of the type adjacency matrix in case of two types of nodes. In Section 6 we prove that, for a class of networks, orders generated by several different centrality measures do coincide**.** In Sections 7 and 8 we prove (by usage of the type adjacency matrix) and discuss conditions of existence of, correspondingly, Katz-Bonacich centrality and $\alpha\gamma$-centrality. Section 9 concludes.

2. Some centrality measures

We start from a reminder of centrality measures, which will be used or mentioned in the paper.

1. Degree – the simplest centrality measure – is the number of neighbors (incident nodes) of a node;

2. Betweenness (or intermediary) centrality. For every pair of nodes in a connected graph, there exists at least one shortest path between the nodes. The betweenness centrality for a node is the number of these shortest paths that pass through the node.

3. Eigenvalue (or eigenvector) centrality. The eigenvalue centrality vector is equal to the Frobenius eigenvector of the adjacency matrix (i.e. the eigenvector corresponding the maximum eigenvalue).

4. Bonacich centrality (Bonacich, 1972). The Bonacich centrality vector is

$$C_B = (I - \alpha A)^{-1} 1,$$

where I is the identity matrix, A is adjacency matrix, 1 is a vector of all ones, α is a positive number. It is shown in Section 7 that the Bonacich centrality exists (all components of the vector C_B are positive) if $0 < \alpha < 1/\lambda_F$, where λ_F is the Frobenius eigenvalue of the adjacency matrix A. Under these conditions, the vector of Bonacich centralities is represented as a sum of a series:

$$C_B = (I + \alpha A + \alpha^2 A^2 + ...)1,$$

which means that the Bonacich centrality of a node measures the discounted number of paths originated from the node.

5. Katz centrality (Katz, 1953, Goyal, 2009) of any node is less by one than the Bonacich centrtality. The vector of Katz centralities is:

$$C_K = ((I - \alpha A)^{-1} - I)1.$$

6. Diffusion centrality (see Banerjee et al., 2013). The concept of diffusion centrality is based on a dynamic contagion process starting at node *i*. In period 1, every neighbor of *i* is contacted with independent probability δ. In period $l = 2$, neighbors of nodes contacted at period $l = 1$ are contacted with independent probability δ. In any arbitrary period *l*, neighbors of nodes contacted at period $l - 1$ are contacted with independent probability *p*. At period *L*, the expected number of times that agents have been contacted is computed using the number of walks:

$$c_i^{dif}(\delta, L) = \sum_{l=1}^{L} \delta^l A^l 1.$$

If $L = 1$, diffusion centrality is proportional to degree centrality. As $L \to \infty$, c_i^{dif} converges to Katz-Bonacich centrality whenever $\delta < 1/\lambda_F$, where λ_F is the Frobenius eigenvalue of adjacency matrix A. Banerjee et al. (2013, 2014) show that diffusion centrality converges to eigenvector centrality as *L* grows whenever $\delta > 1/\lambda_F$.

7. Alpha-centrality (Bonacich, Lloyd, 2001). The vector of α -centralities is

$$C_{\alpha} = (I - \alpha A)^{-1} h ,$$

where I is the identity matrix, A is the adjacency matrix, α is a number (may be negative), h is a vector. Commonly, the vector h is interpreting as a consideration of influences that are independent of the network structure, and the parameter α specifies – as relative weighting of those influences that are induced by the structure of relationships and the exogenous influences. Bonacich (1987) introduced a special case of the alpha centrality, called alpha-beta centrality[2]:

$$C_{\alpha\beta} = \beta(I - \alpha A)^{-1} A1,$$

where I is the identity matrix, 1 is the vector of all ones, β, α are parameters; α may be positive or negative. Another special case of alpha centrality – alpha-gamma centrality –as found by Matveenko and Korolev (2016, 2017), plays a role in a model of knowledge production in network. For an undirected graph described by adjacency matrix A, the vector of alpha-gamma centralities of nodes is

$$C^{\alpha\gamma} = \gamma(I - \alpha A)^{-1} 1,$$

where I is the identity matrix, 1 is the vector of all ones, α and γ are parameters such that $\alpha\gamma < 0$.

8. Decay centrality (see Jackson, 2008) is a measure of distance that takes into account the decay in travelling along shortest paths in the network. It reflects the fact that information traveling along paths in the network may be transmitted stochastically, or that other values or effects transmitted along paths in the network may decay, according to a parameter δ. Decay centrality is defined as

$$c_i^{\delta} = \sum_{l \le n-1} \delta^l n_i^l(A),$$

where $n_i^l(A)$ denote the number of nodes at distance *l* from *i* in network:

$$n_i^l(A) = \left| \left\{ j : \rho_A(i, j) = l \right\} \right|.$$

[2] Bonacich (1987) in his formula uses the letters α, β in the opposite order. We rewrite the definition to obtain a formula of the same type as for the Katz-Bonacich centrality.

As δ goes to 1, decay centrality measures the size of the component in which node *i* lies. As δ goes to 0, decay centrality becomes proportional to degree centrality.

Notice that, while such centrality measures as degree and eigenvector centrality use in their definitions only structural characteristics of network, some other measures, such as Katz-Bonacich centrality and different versions of α-centrality, use the discount factor α (and, maybe, some additional parameters). It may seem that the parameters have an economic sense (preferences and technologies) but not a structural sense, and a choice of them is rather subjective. However, the presence of existence conditions for these measures of centrality considerably diminished arbitrariness of subjective choice of parameters. In particular, the discount factor is in much defined by the network structure. We discuss this question in Section…

3. Types of nodes and typologies of networks

The concept of network typology is based on the fact that the nodes in a network (undirected graph $\langle M, N\rangle$, where $M = \{1,...,n\}$ is a set of nodes, and *N* is a set of edges) may be colored in a minimal number of colors *S* in such way that any node of color *j* (*j*=1,…,S) has definite numbers of neighbors of each of S colors. Such coloring provides a division of the set of nodes *N* into S types.

More formally, a set of types of the nodes of the network is the minimal set *I*={1,…,*S*} for which there exists a mapping $f : M \to I$ such that if $i, j \in M$ and $f(i) = f(j)$, then $f(\nu(i)) = f(\nu(j))$, where $\nu(i)$ is the set of neighbor nodes of node *i*; the set $f(\nu(i)) = \{f(j) : j \in \nu(i)\}$ has $|\nu(i)|$ elements. A polynomial algorithm of division of the set of nodes into types is described in Matveenko, Korolev, 2016, 2017.

To each network a "type adjacency matrix" corresponds. It is a matrix $T = (t_{ij})$ of size $S \times S$, where t_{ij} is the number of neighbors of type *j* for any node of type *i*. Two networks are said to have the same typology if they have the same type adjacency matrix.

A class of one-type networks is the familiar class of regular networks (i.e. such that all nodes have the same degree). The next in the order of complexity is the class of nodes with 2 types of nodes. It is a subclass of the class of networks with two degrees.

Each tree possesses a unique typology. For each network which is not a tree there is a sequence of networks of the same typology with increasing network size. Figure 1 demonstrates two networks of different size with the same typology ($T = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$). Another group of networks with the same typology ($T = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}$) is illustrated in Figure 2.

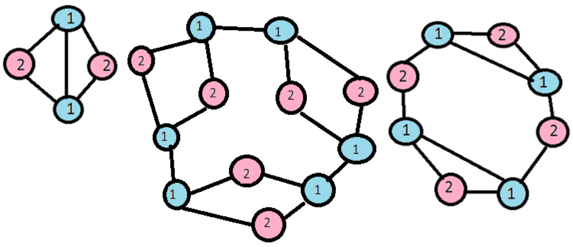


Fig. 1. Networks of different size with the same typology: $T = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$**.**

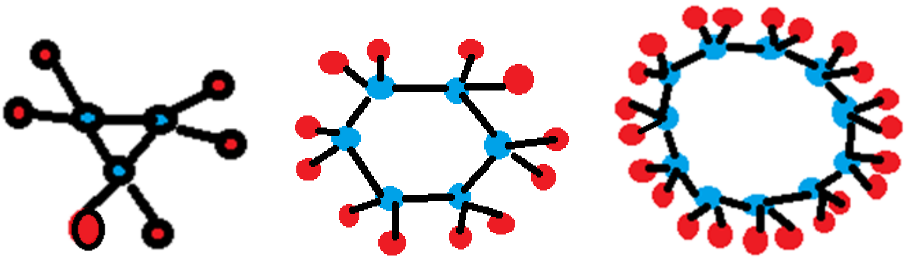


Fig. 2. Networks of different size with the same typology: $T = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}$**.**

Figure 3 shows that networks of similar size (6 nodes) and similar typology ($T = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$) may have different topological structure: in particular, the network in Fig. 3a has a bridge (an edge, after removing of which the number of connected components increases), while there are no bridges in the network in Fig. 3b. The average clusterings are also different: 2/3 for the network in Fig. 3a and 0 for the network in Fig. 3b.

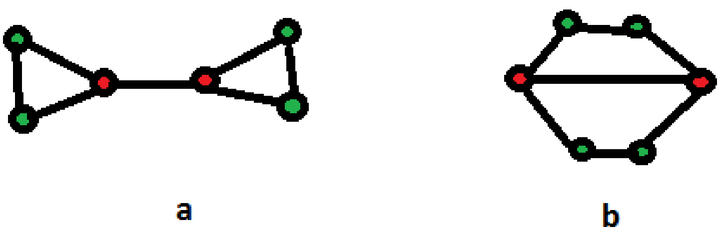


Fig. 3. Networks of the same size and the same typology, but with different topologies.

It is also easy to show that networks of a similar size and a similar distribution of degrees may have different typologies; for example, the network in Figure 4 has four types of nodes: $T = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$, despite it has the same size (6 nodes) and the same distribution of degrees (2 nodes of degree 3 and 4 nodes of degree 2) as the two-type networks shown in Fig. 2.

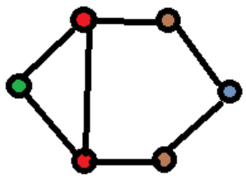


Fig. 4. A 3-type network with the same size and distribution of degrees as in the 2-type networks shown in Fig. 3.

An important structural property of networks is provided by the following Lemma.

Lemma 1. *Let* i, j *be types of nodes, which have, correspondingly,* n_i *and* n_j *nodes. If* $t_{ij} \neq 0$, *then* $t_{ji} \neq 0$ *and*

$$n_i t_{ij} = n_j t_{ji}. \quad (1)$$

Proof. Each of values $n_i t_{ij}, n_j t_{ji}$ express the number of all edges connecting nodes of type i with nodes of type j. □

Networks belonging the same typology possess some common properties defined by the type adjacency matrix T.

Let n_i be the number of type *i* nodes in the network ($i = 1,2,...,S$), m_d be the number of nodes of degree d; $d \in D$, where D is the set of different degrees of nodes present in the network.

Theorem 1. *The following statistics are invariants of typology (they are the same for any networks of given typology):*

1) *relative numbers of nodes of types:* n_i / n_j $(i \neq j;\ i, j = 1,2,...,S)$;

2) *distribution of nodes by types:* $n_i / n \ (i = 1,2,...,S)$*;*

3) *distribution of nodes by degrees:* $m_d / n \ (d \in D)$*;*

4) *relative degree:* $\dfrac{\sum_{d \in D} m_d d}{n}$.

Proof. 1) For types i, j of nodes, if $t_{ij} \neq 0$, then, according to (1), $n_i / n_j = t_{ji} / t_{ij}$, i.e. n_i / n_j is an invariant of typology. If $t_{ij} = 0$, then, because of connectivity, in the network there is a path connecting type i with type j: there exists a finite sequence of numbers of types $i_1 = i, i_2, ..i_{k-1}, i_k = j$, such thato $t_{i_r i_{r+1}} \neq 0$ for all $r = 1,2,...,k-1$. Then property (1) implies

$$\frac{n_i}{n_j} = \frac{n_{i_1}}{n_{i_2}} \cdot \frac{n_{i_2}}{n_{i_3}} \cdot ... \cdot \frac{n_{i_{k-1}}}{n_{i_k}} = \frac{t_{i_2 i_1}}{t_{i_1 i_2}} \cdot \frac{t_{i_3 i_2}}{t_{i_2 i_3}} \cdot ... \cdot \frac{t_{i_k i_{k-1}}}{t_{i_{k-1} i_k}}.$$

Hence, n_i / n_j is an invariant of typology.

2)

$$\frac{n_i}{n} = \frac{n_i}{n_1 + n_2 + ... + n_S} = \frac{1}{1 + \sum_{j=1,...,S;\, j \neq i} \frac{n_j}{n_i}}.$$

By item 1), each of the terms in the denominator in the R.H.S. is an invariant of typology; hence, the whole ration in the R.H.S. is an invariant of typology.

3)

$$\frac{m_d}{n} = \frac{\sum_{i: \sum_{j=1}^{S} t_{ij} = d} n_i}{n_1 + n_2 + ... + n_S} = \frac{\sum_{i: \sum_{j=1}^{S} t_{ij} = d} \frac{n_i}{n_1}}{1 + \frac{n_2}{n_1} + ... + \frac{n_S}{n_1}}.$$

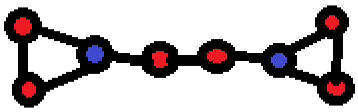
By item 1), each of the terms n_i / n_1 is an invariant of typology; hence, their combination is also an invariant of typology.

4)

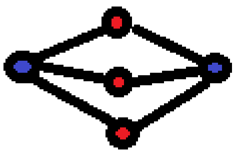
$$\frac{\sum_{d \in D} m_d d}{n} = \sum_{d: \sum_{j=1}^{S} t_{ij} = d \, (i=1,2,...,S)} \frac{m_d}{n} \cdot d.$$

By item 3), each of the terms m_d / n is an invariant of typology; hence, their combination is also an invariant of typology. □

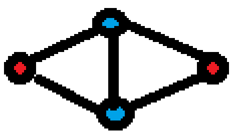
A case of two types of nodes is the nearest generalization of regular (equidegree) networks. The networks with two types of nodes are rather diverse. As an example, Figure 5 demonstrates all 6 typologies which are possible if there are two types of nodes with degrees 3 and 2.



a $T = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}$

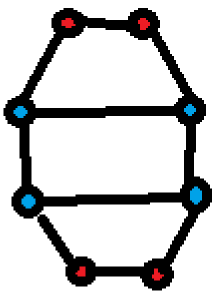


b $T = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$

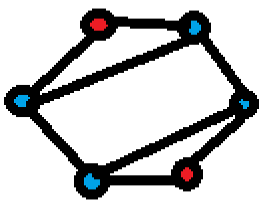


$T = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$

c

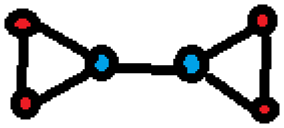


d $T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$



$T = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}$

e



$T = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

f

Figure 5. Six typologies possible under two types of nodes with degrees 3 and 2.

4. Relation of typology and centralities

The following theorem shows that several centrality measures do not just relate to separate nodes of networks, but do correspond to the typology and characterize the types of nodes. The fact that a centrality measure (one or another) is the same for all nodes of the same type, in particular, broadly generalizes a lemma in Acemoglu et al., 2016, in which it is shown that in a regular network Katz-Bonacich centrality is the same for all nodes.

Theorem 2. *In any class of networks with same typology, if i and j are nodes of the same type (may be even belonging different networks of this typology), then c(i)=c(j), where c is any of the following centrality measures: degree, eigenvalue centrality, Katz-Bonacich centrality, diffusion centrality, alpha-gamma centrality, alpha-beta centrality.*

This implies that networks of different size but with the same typology have common properties in economic models; in particular, game equilibria may be transplanted among networks with the same typology.

Theorem 2 is equivalent to the following Theorem 3, which we will prove.

Theorem 3. *For calculation of any of the centrality measures listed in Theorem 2, the type adjacency matrix T may be used instead of the adjacency matrix A.*

Proof. **Degrees.** The degrees may be calculated in similar ways by use of either the adjacency matrix, A, or the type adjacency matrix, T. The degrees of node i ($i \in 1,2,,...,n$) and of any node of type $\tilde{i}$ ($\tilde{i} \in 1,2,...,S$) are, correspondingly,

$$d_i = \sum_{j=1}^{n} a_{ij},$$

$$d_{\tilde{i}} = \sum_{\tilde{j}=1}^{S} t_{\tilde{i}\tilde{j}}.$$

If $i \in N_{\tilde{i}}$, then $d_i = d_{\tilde{i}}$. In matrix form: $d = A1,\ \tilde{d} = T\tilde{1},$ where $\tilde{d}$ is the *S*-vector of degrees for types of nodes, and $\tilde{1}$ is the *S*-vector of all ones.

Katz-Bonacich centrality. Since only the summary discounted numbers of walks from node *i* are needed in calculation of both Katz-Bonacich centrality measures, the type adjacency matrix T can be used instead of the adjacency matrix A. We obtain the *S*-vector of Bonacich centrality measures for types:

$$\tilde{C}^B = (\tilde{I} - \alpha T)^{-1}\tilde{1},$$

where $\tilde{I}$ is the $S \times S$ identity matrix, and $\tilde{1}$ is the S-vector of all ones; $0 < \alpha < 1$.

Each degree matrix T^k consists of the elements $t^k_{\tilde{i}\tilde{j}} = \sum_{j=1}^{S} a^k_{ij}$, where a^k_{ij} are the elements of the degree matrix A^k (k=1,2,…); $i \in N_{\tilde{i}}$ is any node of type $\tilde{i}$. Correspondingly,

$$\tilde{C}^B = \tilde{I} + \alpha\tilde{T} + \alpha^2 T^2 + ...$$

Similarly, the vector of Katz centrality measures for types is

$$\tilde{C}^K = ((\tilde{I} - \alpha T)^{-1} - \tilde{I})\tilde{1}.$$

Eigenvalue centrality. It follows directly from the definitions that (i) if λ is an eigenvalue of the type adjacency matrix T and $\tilde{b}$ is a corresponding eigenvector, then λ is also an eigenvalue of the adjacency matrix A, and the corresponding eigenvector is b, where $b_i = \tilde{b}_{\tilde{i}}$ if $i \in N_{\tilde{i}}$. (In other words, $\lambda\tilde{b} = T\tilde{b}$ implies $\lambda b = Ab$.) Let us proof that (ii) if λ_F is the Frobenius eigenvalue of the type adjacency matrix T, then λ_F is also the Frobenius eigenvalue of the adjacency matrix A.

To prove (ii) ad absurdum, assume that the Frobenius eigenvalue of matrix A is $\mu \neq \lambda_F$. Part (i) implies that $\mu > \lambda_F$. Let *e* be the Frobenius eigenvector of matrix A and let $\hat{e}$ be the *n*-vector with components

$$\hat{e}_i = \max_{j \in N_i} e_j,\quad i = 1,2,...,n.$$

Evidently,

$$A\hat{e} \geq \mu e.$$

Let $\hat{f}$ be an S-vector corresponding to $\hat{e}$, i.e. $\hat{f}_{\tilde{i}} = \hat{e}_i$ if $i \in N_{\tilde{i}}$. Then

$$T\hat{f} \geq \mu\hat{f}. \qquad (2)$$

But, according to the Perron-Frobenius theorem since λ_F is the Frobenius eigenvalue, (2) implies $\lambda_F \geq \mu$. Contradiction!

Alpha-gamma centrality. It is easy to show that the vector of $\alpha\gamma$ -centralities

$$C^{\alpha\gamma} = \gamma(I - \alpha A)^{-1}1$$

where α, γ are numbers, such that $\alpha\gamma < 0$, if it exists (is positive), can be calculated for types by use of the type adjacency matrix T:

$$\tilde{C}^{\alpha\gamma} = \gamma(\tilde{I} - \alpha T)^{-1}\tilde{1}.$$

Here $\tilde{C}^{\alpha\gamma}$ is the *S*-vector of $\alpha\gamma$ -centralities for types, $\tilde{I}$ is the $S \times S$ identity matrix, $\tilde{1}$ is the S-vector of all ones.

Alpha-beta centrality. It is easy to show that the vector of centralities

$$C_{\alpha\beta} = \beta(I - \alpha A)^{-1}A1 \qquad (3)$$

where α, β are numbers, if it exists (is positive), can be calculated for types by use of the type adjacency matrix T:

$$\tilde{C}_{\alpha\beta} = \beta(\tilde{I} - \alpha T)^{-1}T\tilde{1}. \qquad (4)$$

Here $\tilde{C}_{\alpha\beta}$ is the *S*-vector of $\alpha\beta$-centralities for types, $\tilde{I}$ is the $S \times S$ identity matrix, $\tilde{1}$ is the S-vector of all ones.

Let the centralities vector $C_{\alpha\beta}$ exist. It implies that the matrix $I - \alpha A$ is nondegenerate, and the vector $C_{\alpha\beta}$, defined by (3), is strictly positive and is a unique solution of equation

$$(I - \alpha A)C_{\alpha\beta} = \beta A1. \qquad (5)$$

The nondegeneracy of matrix $I - \alpha A$ means, that among the eigenvalues of matrix A there is no $1/\alpha$. But then also $1/\alpha$ can not be an eigenvalue of matrix *T* (we have shown above that any eigenvalue of *T* is also an eigenvalue of *A*). Hence, matrix $\tilde{I} - aT$ is also nondegenerate, and there exists a vector $\tilde{C}_{\alpha\beta}$, defined by (4). This vector is a unigue solution of the following equation

$$(\tilde{I} - \alpha T)\tilde{C}_{\alpha\beta} = \beta T\tilde{1}. \qquad (6)$$

It is clear from comparison of equations (5) and (6) that to each component $\tilde{C}_{\alpha\beta}(i)$ of vector $\tilde{C}_{\alpha\beta}$, which expresses the $\alpha\beta$-centrality of type типа *i*, n_i equal components of vector $C_{\alpha\beta}$ correspond, which express $\alpha\beta$-centrality of each of n_i nodes of type *i*.

5. Mathematical properties of the type adjacency matrix in case of two types of nodes

As was already said, the class of networks with two types of nodes is of a special interest. In this Section we derive some important properties of the type adjacency T in case of two types of nodes. These properties will be used further in Section 8.

The eigenvalues of the type adjacency matrix are

$$\lambda = \frac{t_{11} + t_{22} \pm \sqrt{(t_{11} - t_{22})^2 + 4t_{12}t_{21}}}{2}, \qquad (7)$$

which is the same as

$$\lambda = \frac{TrT \pm \sqrt{(TrT)^2 - 4DetT}}{2},$$

where $TrT, DetT$ are, correspondingly trace and determinant of matrix T. We will denote the Frobenius eigenvalue by λ_F and the second eigenvalue by λ_2. By definition of Frobenius eigenvalue, $|\lambda_F| > |\lambda_2|$. In our case, evidently, $\lambda_F > 0$.

Lemma 2.

$$\lambda_F \geq \sqrt{2},$$

$$\lambda_2 > 0 \Leftrightarrow DetT > 0,$$

$$\lambda_2 > 1 \Leftrightarrow DetT > TrT - 1.$$

Proof. Because of connectedness, $t_{12} \geq 1, t_{21} \geq 1$. But the case of $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is impossible, as far as we consider a two-type network (but not a regular network with only one type of nodes). It is clear from (7), that the minimal feasible λ_F is in case of $t_{12}t_{21} \geq 2$; hence, $\lambda_F \geq \sqrt{2}.$

Inequality $\lambda_2 > 0$ is equivalent to $TrT > \sqrt{(TrT)^2 - 4DetT}$, which implies $DetT > 0.$

Inequality $\lambda_2 > 1$ is equivalent to $TrT - 2 > \sqrt{(TrT)^2 - 4DetT}$, which reduces to $DetT > TrT - 1.$ □

Lemma 3.

$$\min\{t_{11} - t_{21}, t_{22} - t_{12}\} \leq \lambda_2 \leq \max\{t_{11} - t_{21}, t_{22} - t_{12}\} \leq \lambda_F.$$

Proof. It is sufficient to consider the case when type 1 has the biggest degree:

$t_{11} + t_{12} \geq t_{21} + t_{22}$, that is

$$t_{11} - t_{21} \geq t_{22} - t_{12}. \quad (8)$$

(In the opposite case only the indices of types change, so the statement of the Lemma stays in power).

1. We have to prove that $t_{11} - t_{21} \leq \lambda_F$. Assume the opposite: $\lambda_F < t_{11} - t_{21}$. Then

$$t_{11} + t_{22} + \sqrt{(t_{11} - t_{22})^2 + 4t_{12}t_{21}} < 2(t_{11} - t_{21}),$$

which implies

$$\sqrt{(t_{11} - t_{22})^2 + 4t_{12}t_{21}} < (t_{11} - t_{22}) - 2t_{21}.$$

After raising to square and reduction, we have:

$$4t_{12}t_{21} < -4t_{21}(t_{11} - t_{22}) + 4t_{21}^2,$$

which reduces to $t_{11} - t_{21} < t_{22} - t_{12}$. This contradicts inequality (8).

2. We prove that $\lambda_2 \leq t_{11} - t_{21}.$ Assume the opposite: $\lambda_2 > t_{11} - t_{21}.$ Then

$$t_{11} + t_{22} - \sqrt{(t_{11} - t_{22})^2 + 4t_{12}t_{21}} > 2(t_{11} - t_{21}),$$

i.e.

$$\sqrt{(t_{11}-t_{22})^2+4t_{12}t_{21}} < (t_{22}-t_{11})+2t_{21}.$$

After raising to square and reduction, we have the same inequality as in case 1:

$$4t_{12}t_{21} < 4t_{21}(t_{22}-t_{11})+4t_{21}^2,$$

and, again, it leads to a contradiction.

3. We prove that $t_{22}-t_{12} \le \lambda_2$. Assume the opposite: $\lambda_2 < t_{22}-t_{12}$. Then

$$t_{11}+t_{22}-\sqrt{(t_{11}-t_{22})^2+4t_{12}t_{21}} < 2(t_{22}-t_{12}),$$

i.e.

$$\sqrt{(t_{11}-t_{22})^2+4t_{12}t_{21}} > t_{11}-t_{22}+2t_{12}. \qquad (9)$$

If the R.H.S. of inequality (9) is nonpositive, then

$$t_{12}+t_{11} \le t_{22}-t_{12} \le t_{11}-t_{21},$$

this implies

$$t_{12}+t_{21} \le 0,$$

which contradicts connectedness. Hence, the R.H.S. of inequality (9) has to be positive. After raising to square and reduction, we have

$$4t_{12}t_{21} > 4t_{12}(t_{11}-t_{22})+4t_{12}^2,$$

which leads to inequality

$$t_{22}-t_{12} > t_{11}-t_{21},$$

which contradicts inequality (8). □

Lemma 4. *The determinant of matrix* $(I-\alpha T)$

is positive

$$\Delta > 0,\ if\ \frac{1}{\alpha} < \lambda_2\ \ or\ \ \frac{1}{\alpha} > \lambda_F,$$

is negative

$$\Delta < 0\,,\ if\ \lambda_2 < \frac{1}{\alpha} < \lambda_F\,.$$

Proof. We see, that $\Delta = 1 - \alpha TrT + \alpha^2 DetT$ has the same sign as the sign of function $\chi^2 - \chi TrT + DetT$ in point $\chi = \frac{1}{\alpha}$. The latter function is a characteristic polynomial of matrix T; its roots are λ_2 and λ_F; this polynomial is negative on the interval (λ_2, λ_F) and is positive to both sides of this interval. □

6. Classes of networks, for which orders generated by several different centrality measures do coincide

König et al., 2014 and Bloch et al., 2017 formulate the following problem: for which classes of networks each of a set of several centrality measures defines the same order on the set of nodes of network? (Here are two questions: about the class of networks and about the set of centrality measures). In particular, Bloch et al., 2017 introduce a class of trees – so called regular monotonous hierarchies – and prove (Proposition 2 and Corollary 1 in their paper) that, for any tree of this class, orders defined on the set of nodes by the following centrality measures: degree centrality, decay centrality, Katz-Bonacich centrality, diffusion centrality, intermediary centrality do coincide. For each tree not belonging the class, there exists a pair of centralities listed above which defines different orders on the set of nodes.

Applying to undirected networks, the class of regular monotonous hierarchies might be introduced in the following way. Let $\rho(i)$ be a distance between a node *i* and a root of the tree, and d_i be degree of node *i*.

Definition 1. Tree *g* is called *regular monotonous hierarchy* if there exists a node i_0 (root) such that the tree satisfies following conditions:

- All nodes being at the same distance from the root have the same degree (i.e. if $\rho(i) = \rho(j)$, then $d_i = d_j$).
- For any two nodes *i, j,* if the distance between the root and *i,* $\rho(i)$ is less than the distance between the root and *j,* $\rho(j)$, then $d_i \geq d_j$.

Evidently, all leafs are at the same distance from the root. An example of regular monotonous hierarchy is a tree presented in Figure 6. Bloch et al., 2017 prove the following statement.

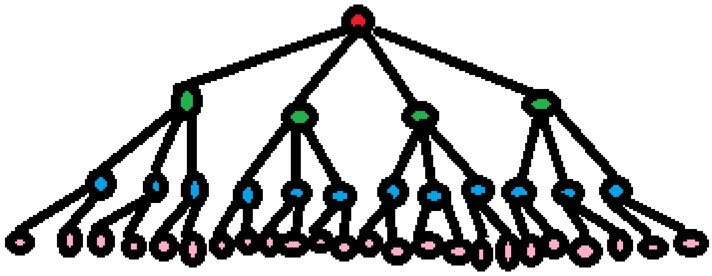


Fig. 6. An example of regular monotonous hierarchy.

Theorem 4 (Proposition 2, Corollary 1 in Bloch et al., 2017)**.** *If a tree g is a regular monotonous hierarchy, then the following centrality measures:*

1) *degree centrality;*
2) *decay centrality;*
3) *Katz-Bonacich centrality;*
4) *diffusion centrality;*
5) *intermediary centrality*

order the nodes in the same way. For each tree not of the class of regular monotonous hierarchies, there is a pair of centralities from this list, which will give different orders on the set of nodes.

We show that such kind classes are not limited by trees.

Theorem 5. *For any network with two types of nodes, the orders defined on the set of nodes by the following centrality measures:*

1) *degree;*
2) *eigenvalue centralit;*
3) *Katz-Bonacich centrality (under condition of its existence);*
4) *diffusion centrality;*
5) *alpha-gamma centrality (under condition of existence and* $\Delta > 0$*)*
6) *alpha-beta centrality (under condition of existence and an additional condition)*

do coincide.

Proof. **1. Degrees and eigenvector centrality.** The vector $\tilde{C}^e$ of eigenvector centralities is defined by the equation

$$T\tilde{C}^e = \alpha_F \tilde{C}^e,$$

which implies

$$t_{11}\tilde{C}_1^e + t_{12}\tilde{C}_2^e = \frac{t_{11}+t_{22}+\sqrt{(t_{11}-t_{22})^2+4t_{12}t_{21}}}{2}\tilde{C}_1^e.$$

Hence,

$$t_{12}\tilde{C}_2^e = \left(\frac{t_{11}+t_{22}+\sqrt{(t_{11}-t_{22})^2+4t_{12}t_{21}}}{2} - t_{11}\right)\tilde{C}_1^e,$$

$$\tilde{C}_1^e > \tilde{C}_2^e \iff 2t_{12}+t_{11}-t_{22} > \sqrt{(t_{11}-t_{22})^2+4t_{12}t_{21}}$$

$$\iff 4t_{12}^2 + 4t_{12}(t_{11}-t_{22}) > 4t_{12}t_{21} \iff d_1 > d_2.$$

2. Degrees and Bonacich centrality. Let us proof that

$$\tilde{C}_1^B - \tilde{C}_2^B = \frac{1}{\alpha(\frac{1}{\alpha}-\lambda_F)(\frac{1}{\alpha}-\lambda_2)}(d_1-d_2). \tag{10}$$

Indeed,

$$\tilde{C}_1^B - \tilde{C}_2^B = \frac{1}{\Delta}[1+\alpha(t_{12}-t_{22})-1-\alpha(t_{21}-t_{11})] = \frac{\alpha}{\Delta}(d_1-d_2),$$

$$\Delta = 1-\alpha TrT + \alpha^2 DetT = \alpha^2\left(\frac{1}{\alpha^2}-\frac{1}{\alpha}TrT + DetT\right) = \alpha^2(\frac{1}{\alpha}-\lambda_F)(\frac{1}{\alpha}-\lambda_2).$$

The Bonacich centrality exists iff $\frac{1}{\alpha} > \lambda_F \geq \lambda_2$; hence, as it is seen from (10), the orders of $\tilde{C}_1^B, \tilde{C}_2^B$ and d_1, d_2 do coinside.

4. **Degrees and diffusion centrality.** We will use the definition of diffusion centrality with a free term:

$$\left(I + \alpha T + \alpha^2 T^2 + \ldots + \alpha^L T^L\right)\tilde{1}$$

(Diffusion centralities without free term differ by one in any node).

If $L = 1$, then the values of diffusion centrality of types are

$$C_{\tilde{i}}^{dif} = 1 + \alpha d_{\tilde{i}}.$$

The value α is positive, thus, if $L = 1$, then the diffusion centralities induce on the set of nodes the same order as degrees of nodes.

Let us show that in a network with two types of nodes, the order of nodes induced by diffusion centrality does not depend on the natural number *L*.

Remind that for the type adjacency matrix,

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix},$$

the eigenvalues are

$$\lambda_F = \frac{t_{11} + t_{22} + \sqrt{(t_{11} - t_{22})^2 + 4t_{12}t_{21}}}{2}, \; \lambda_2 = \frac{t_{11} + t_{22} - \sqrt{(t_{11} - t_{22})^2 + 4t_{12}t_{21}}}{2}.$$

Let

$$\Lambda = \begin{pmatrix} \lambda_F & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

and let C be a matrix of transition to a basis constructed from the eigenvectors of matrix *T*; i.e. columns of matrix C are eigenvectors corresponding eigenvalues λ_F and λ_2. Elements of matrix C will be denoted c_{ij}, and elements of the inverse matrix C^{-1} will be denoted $\tilde{c}_{ij}$. Then

$$\left(I + \alpha T + \alpha^2 T^2 + \ldots + \alpha^L T^L\right)\tilde{1} = C\left(I + \alpha\Lambda + \alpha^2\Lambda^2 + \ldots + \alpha^L\Lambda^L\right)C^{-1}\tilde{1} =$$

$$= \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \frac{1-\lambda_F^{L+1}}{1-\lambda_F} & 0 \\ 0 & \frac{1-\lambda_2^{L+1}}{1-\lambda_2} \end{pmatrix} \begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} \\ \tilde{c}_{21} & \tilde{c}_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} c_{11}\frac{1-\lambda_F^{L+1}}{1-\lambda_F} & c_{12}\frac{1-\lambda_2^{L+1}}{1-\lambda_2} \\ c_{21}\frac{1-\lambda_F^{L+1}}{1-\lambda_F} & c_{22}\frac{1-\lambda_2^{L+1}}{1-\lambda_2} \end{pmatrix} \begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} \\ \tilde{c}_{21} & \tilde{c}_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} c_{11}\tilde{c}_{11}\frac{1-\lambda_F^{L+1}}{1-\lambda_F} + c_{12}\tilde{c}_{21}\frac{1-\lambda_2^{L+1}}{1-\lambda_2} & c_{11}\tilde{c}_{12}\frac{1-\lambda_F^{L+1}}{1-\lambda_F} + c_{12}\tilde{c}_{22}\frac{1-\lambda_2^{L+1}}{1-\lambda_2} \\ c_{21}\tilde{c}_{11}\frac{1-\lambda_F^{L+1}}{1-\lambda_F} + c_{22}\tilde{c}_{21}\frac{1-\lambda_2^{L+1}}{1-\lambda_2} & c_{21}\tilde{c}_{12}\frac{1-\lambda_F^{L+1}}{1-\lambda_F} + c_{22}\tilde{c}_{22}\frac{1-\lambda_2^{L+1}}{1-\lambda_2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{11}\tilde{c}_{11}\frac{1-\lambda_F^{L+1}}{1-\lambda_F} + c_{12}\tilde{c}_{21}\frac{1-\lambda_2^{L+1}}{1-\lambda_2} + c_{11}\tilde{c}_{12}\frac{1-\lambda_F^{L+1}}{1-\lambda_F} + c_{12}\tilde{c}_{22}\frac{1-\lambda_2^{L+1}}{1-\lambda_2} \\ c_{21}\tilde{c}_{11}\frac{1-\lambda_F^{L+1}}{1-\lambda_F} + c_{22}\tilde{c}_{21}\frac{1-\lambda_2^{L+1}}{1-\lambda_2} + c_{21}\tilde{c}_{12}\frac{1-\lambda_F^{L+1}}{1-\lambda_F} + c_{22}\tilde{c}_{22}\frac{1-\lambda_2^{L+1}}{1-\lambda_2} \end{pmatrix}. \qquad (11)$$

Hence, the difference of the 1st and 2nd types nodes diffusion centralities (the difference of 1st and 2nd rows of matrix (1)) is

$$Dif = \frac{1-\lambda_F^{L+1}}{1-\lambda_F}\left(\tilde{c}_{11}+\tilde{c}_{12}\right)\left(c_{11}-c_{21}\right)+\frac{1-\lambda_2^{L+1}}{1-\lambda_2}\left(\tilde{c}_{21}+\tilde{c}_{22}\right)\left(c_{12}-c_{22}\right). \qquad (12)$$

The formula (12) is true for any natural L, and also for $L=0$. But for $L=0$, obviously, $Dif=0$; hence,

$$(\tilde{c}_{11}+\tilde{c}_{12})(c_{11}-c_{21})+(\tilde{c}_{21}+\tilde{c}_{22})(c_{12}-c_{22})=0.$$

If degrees of the 1st type nodes are higher than of the 2nd type nodes, then for $L=1$ we have $Dif>0$, which implies

$$\left(\tilde{c}_{11}+\tilde{c}_{12}\right)\left(c_{11}-c_{21}\right)>0,\ \left(\tilde{c}_{21}+\tilde{c}_{22}\right)\left(c_{12}-c_{22}\right)<0.$$

But then the same sign the difference of diffusion centralities Dif has for any other natural $L>1$, as

$$\frac{1-\lambda_F^{L+1}}{1-\lambda_F}=1+\lambda_F+\lambda_F^2+\ldots+\lambda_F^L,\ \frac{1-\lambda_2^{L+1}}{1-\lambda_2}=1+\lambda_2+\lambda_2^2+\ldots+\lambda_2^L,$$

$$1+\lambda_F+\lambda_F^2+\ldots+\lambda_F^L>0,\ 1+\lambda_F+\lambda_F^2+\ldots+\lambda_F^L>\left|1+\lambda_2+\lambda_2^2+\ldots+\lambda_2^L\right|,$$

i.e. the sign of (12) is defined by the sign of $\left(\tilde{c}_{11}+\tilde{c}_{12}\right)\left(c_{11}-c_{21}\right)$:

if $\left(\tilde{c}_{11}+\tilde{c}_{12}\right)\left(c_{11}-c_{21}\right)>0$, then $Dif>0$ for any L;

if $\left(\tilde{c}_{11}+\tilde{c}_{12}\right)\left(c_{11}-c_{21}\right)<0$, then $Dif<0$ for any L.

4. Degrees and $\alpha\gamma$ centrality. The vector of $\alpha\gamma-$centralities of types is

$$\tilde{C}^{\alpha\gamma}=\frac{\gamma}{\Delta}\begin{pmatrix}1+\alpha(t_{12}-t_{22}\\ 1+\alpha(t_{21}-t_{11}\end{pmatrix}=\begin{pmatrix}\frac{\gamma}{\Delta}+\frac{\alpha\gamma}{\Delta}(t_{12}-t_{22})\\ \frac{\gamma}{\Delta}+\frac{\alpha\gamma}{\Delta}(t_{21}-t_{11})\end{pmatrix}.$$

As $\alpha\gamma>0$, if $\Delta>0$, then $\tilde{C}_1^{\alpha\gamma}>\tilde{C}_2^{\alpha\gamma}\ \Leftrightarrow\ t_{12}-t_{22}>t_{21}-t_{11}\ \Leftrightarrow\ d_1>d_2$. If $\Delta<0$, then, vice versa, $\tilde{C}_1^{\alpha\gamma}>\tilde{C}_2^{\alpha\gamma}\ \Leftrightarrow\ d_1<d_2$.

5. Degrees and alpha-beta centrality.

We have

$$\tilde{I}-\alpha T=\begin{pmatrix}1-\alpha t_{11} & -\alpha t_{12}\\ -\alpha t_{21} & 1-\alpha t_{22}\end{pmatrix},$$

$$\Delta=1-\alpha t_{11}-\alpha t_{22}+\alpha^2 t_{11}t_{22}-\alpha^2 t_{12}t_{21},$$

$$\beta\left(\tilde{I}-\alpha T\right)^{-1}=\frac{\beta}{\Delta}\begin{pmatrix}1-\alpha t_{22} & \alpha t_{12}\\ \alpha t_{21} & 1-\alpha t_{11}\end{pmatrix},$$

$$\beta\left(\tilde{I}-\alpha T\right)^{-1}T=\frac{\beta}{\Delta}\begin{pmatrix}t_{11}-\alpha t_{11}t_{22}+\alpha t_{12}t_{21} & t_{12}\\ t_{21} & \alpha t_{12}t_{21}+t_{22}-\alpha t_{11}t_{22}\end{pmatrix},$$

$$\tilde{C}_{\alpha\beta}=\beta\left(\tilde{I}-\alpha T\right)^{-1}T1=\frac{\beta}{\Delta}\begin{pmatrix}t_{11}+t_{12}-\alpha\left(t_{11}t_{22}-t_{12}t_{21}\right)\\ t_{21}+t_{22}-\alpha\left(t_{11}t_{22}-t_{12}t_{21}\right)\end{pmatrix}.$$

Thus, the $\alpha\beta$-centrality exists if

$$\beta\Delta>0 \text{ и } t_{11}+t_{12}>\alpha\left(t_{11}t_{12}-t_{12}t_{21}\right),\ t_{21}+t_{22}>\alpha\left(t_{11}t_{22}-t_{12}t_{21}\right)$$

or

$$\beta\Delta<0 \text{ и } t_{11}+t_{12}<\alpha\left(t_{11}t_{12}-t_{12}t_{21}\right),\ t_{21}+t_{22}<\alpha\left(t_{11}t_{22}-t_{12}t_{21}\right).$$

In the former case, the $\alpha\beta$-centrality defines on the set of nodes the same order as degrees, and in the latter case – the opposite order. □

7. Condition of existence of Bonacich centrality

Opposite to many other centrality measures, the Katz-Bonacich and the alpha-gamma centralities exist not always, but only in some definite regions of parameter α. The Bonacich centralities are positive iff $0<\alpha<1/\lambda_F$, where λ_F is the Frobenius eigenvalue of any of matrices T and A. This existence conditioned is mentioned occasionally by many authors; however, it is difficult to refer to its full proof. We provide a brief proof of the fact and discuss it. Notice that the possibilities to subjectively choose parameter α are limited, since λ_F is defined by the network. For example, for the star network, $1/\lambda_F\to 0$ as $n\to\infty$, so α has to has to become unlimitedly small when the network increases.

Theorem 6. *The vectors of Bonacich centralities for nodes and for types exist iff* $\alpha<\frac{1}{\lambda_F}$.

Proof. The Bonacich centrality exists if the matrix $I-\alpha T$ is invertible and the vector $\left(I-\alpha T\right)^{-1}\tilde{1}$ is strictly positive. We will prove that this takes place iff $\alpha<\frac{1}{\lambda_F}$, i.e. iff the spectral radius of matrix αT is less than 1.

It is known (see e.g. Debreu, Herstein, 1953) that for any square matrix T with nonnegative elements, the matrix $I-T$ is invertible and

$$(I-T)^{-1}=I+T+T^2+...=\sum_{k=0}^{\infty}T^k \qquad (13)$$

iff the spectral radius of matrix T is less than 1. Thus, to one side our statement is evident: if $\alpha\lambda_F<1$, then the Bonacich centrality does exist. Let us prove the inverse statement.

Proposition 1. Let T be a matrix with nonnegative elements, and let exist such strictly positive vector $\tilde{X}$, that $(I-T)\tilde{X}=\tilde{1}$.Then the matrix $I-T$ is invertible and (13) takes place (i.e. the spectral radius of matrix A is less than 1).

Proof. If $(I-T)\tilde{X}=\tilde{1}$ for $\tilde{X}>>0$, then $T\tilde{X}<<\tilde{X}$, i.e. $T\tilde{X}\le\lambda\tilde{X}$ for a number $\lambda\in(0,1)$. Then $T^2\tilde{X}\le\lambda T\tilde{X}\le\lambda^2\tilde{X}$ and so on. We obtain $T^k\tilde{X}\le\lambda^k\tilde{X}$. But $\lambda^k\underset{k\to\infty}{\to}0$, thus $T^k\underset{k\to\infty}{\to}0$. But $\tilde{X}>>0$ implies that $T^k\underset{k\to\infty}{\to}0$ elementwise. Moreover, $\sum_{k=0}^{\infty}T^k$ converges, since each element of this matrix series is majorated by a convergent geometrical progression. Denote

$$B=\sum_{k=0}^{\infty}T^k;\ B_k=\sum_{i-0}^{k}T^i.$$

We have

$$(\tilde{I}-T)B_k=B_k(\tilde{I}-T)=(\tilde{I}-T)(\tilde{I}+T+T^2+...+T^k)=$$

$$=\tilde{I}-T+T-T^2+T^2-....-T^k+T^k-T^{k+1}=\tilde{I}-T^{k+1}\underset{k\to\infty}{\to}I.$$

Thus,

$$\lim_{k\to\infty}(\tilde{I}-T)B_k=\lim_{k\to\infty}B_k(\tilde{I}-T)=(\tilde{I}-T)B=B(\tilde{I}-T)=\tilde{I},$$

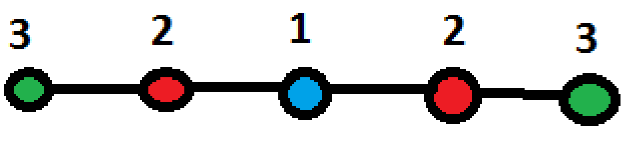
i.e.

$$B=\sum_{k=1}^{\infty}T^k=(\tilde{I}-T)^{-1}.$$

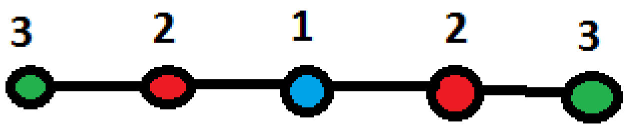
□ □

Examples. Let us provide several examples of conditions for existence of Bonacich centralities for different networks.

For chain networks with 3 types of nodes, shown in Figure 7, the conditions of existence of Bonacich centrality are, correspondingly, inequalities a) $0<\alpha<0.58$ and b) $0<\alpha<0.55$.



a) $0 < \alpha < 0.58$



b) $0 < \alpha < 0.55$.

Fig. 7. Conditions of existence of Bonacich centrality for 2 chain networks with 3 types of nodes.

For any network with two types of nodes, Lemma 2 provides an estimate for feasible α:

$$\alpha < \frac{1}{\lambda_F} \leq \frac{1}{\sqrt{2}} \approx 0.71.$$

For typologies with two types of nodes with degrees 3 and 2, shown in Figure 5 (cases a,b,c,d,e,f), the conditions of existaence of Bonacich centrality give much more restrictive inequalities on parameter α:

a) $0 < \alpha < \frac{\sqrt{13}-1}{6} \approx 0.43$

b) $0 < \alpha < \frac{1}{\sqrt{6}} \approx 0.41$

c) $0 < \alpha < \frac{\sqrt{17}-1}{8} \approx 0.39$

d) $0 < \alpha < \frac{3-\sqrt{5}}{2} \approx 0.38$

e) $0 < \alpha < \frac{\sqrt{3}-1}{2} \approx 0.36$

f) $0 < \alpha < \frac{1}{3} \approx 0.33$

If, acting in a definite network, agents used to use a definite discount factor α, then after a change in the structure of the network, they may face a necessity to diminish the discount factor to ensure a solution. Thus, the discount factor α is not fully subjective, but in considerable degree it is defined by the network structure.

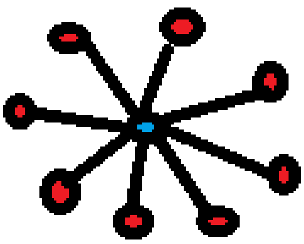


Fig 8. Star network.

A case of star network is representative. For a star network with ν peripheral nodes, the condition of existence of Bonacich centrality is the inequality $0<\alpha<\frac{1}{\nu}$. The discount factor α has to become arbitrarily small when the star network grows! A decision maker has no possibility to choose the discount factor in such case!

Katz-Bonacich centrality and α-centrality are based on a use of a discount factor α, which is traditionally thought as having an economic, but not a network-structural sense (it is similar to the discount factor in dynamic economic models). However, we show that parameter α may not be arbitrary subjective. For exampole, in the star network, if the number of the rays grows, for the economic model leading to the Bonacich centrality to continue to work, parameter α has to arbitrarily diminish, i.e. the agents have to become "infinitely impatient" in comparison of their close friends and friends of friends.

8. Conditions of existence of $\alpha\gamma$-centrality under presence of two types of nodes

Bonacich centrality and $\alpha\gamma$-centrality never coexist under a joint α. Indeed, the vector of $\alpha\gamma$-centralities is

$$C^{\alpha\gamma}=\gamma(I-\alpha T)^{-1}1=\frac{\gamma}{\Delta}\begin{pmatrix}1+\alpha(t_{12}-t_{22})\\1+\alpha(t_{21}-t_{11})\end{pmatrix},\quad (14)$$

where Δ is the determinant of matrix $(I-\alpha T)$. It is easy to see that $\alpha\gamma$-centralities do not exist under condition of existence of the Bonacich centrality (otherwise the equality $C^{\alpha\gamma}=\gamma C^{B}$ would be checked under $\gamma<0$, which is impossible).

Let us introduce notations

$$m = \min\{t_{22} - t_{12}, t_{11} - t_{21}\},\ M = \max\{t_{22} - t_{12}, t_{11} - t_{21}\}.$$

Let us suppose, for definiteness, that the 1st type nodes gave a higher degree than the 2nd type:

$$t_{11} + t_{12} > t_{21} + t_{22}$$

(the equality of degrees is impossible, since it would mean that there is only one type). Then

$$m = t_{22} - t_{12},\ M = t_{11} - t_{21}.$$

Theorem 7. 1. *If* $m > 0$, *then the condition of existence of* $\alpha\gamma$ *-centrality is*

$$\alpha \in (-\infty, 0) \cup (\frac{1}{\lambda_F}, \frac{1}{M}) \cup (\frac{1}{m}, +\infty).$$

2. If $m = 0$, *then the condition of existence of* $\alpha\gamma$ *-centrality is*

$$\alpha \in (-\infty, 0) \cup (\frac{1}{\lambda_F}, \frac{1}{M})..$$

3. If $M < 0$, *then the condition of existence of* $\alpha\gamma$ *-centrality is*

$$\alpha \in (-\infty, \frac{1}{M}) \cup (\frac{1}{m}, 0) \cup (\frac{1}{\lambda_F}, +\infty).$$

\4. If $M = 0$, *then the condition of existence of* $\alpha\gamma$ *-centrality is*

$$\alpha \in (\frac{1}{m}, 0) \cup (\frac{1}{\lambda_F}, +\infty).$$

5. If $m < 0,\ M > 0$, *then the condition of existence of* $\alpha\gamma$ *-centrality is*

$$\alpha \in (\frac{1}{m}, 0) \cup (\frac{1}{\lambda_F}, \frac{1}{M}).$$

Proof. 1-2. Inequalities $t_{22} \geq t_{12}, t_{11} > t_{21}$ imply $DetT > 0$; hence, $\lambda_2 > 0$.

If $\alpha < 0$, then $\gamma > 0$ and, as is seen from formula (14), for existence of $\alpha\gamma$ -centrality it is needed $\Delta > 0$ (as $1 + \alpha(t_{12} - t_{22}) > 0,\ 1 + \alpha(t_{21} - t_{11}) > 0$). Then, by Lemma 3, it is demanded

$\frac{1}{\alpha} < \lambda_2$ (accounting for $\alpha < 0$). But this condition is satisfied automatically, i.e. $\alpha\gamma$ -centrality exists for $\alpha < 0$, .

If $\alpha > 0,\ \gamma < 0,$ then $\gamma < 0$, and $\alpha\gamma$ -centrality may exist both under $\Delta > 0$ and under $\Delta < 0$.

If $1+\alpha(t_{12}-t_{22}) < 0,\ 1+\alpha(t_{21}-t_{11}) < 0,$ what means $\frac{1}{\alpha} < m$ (here $m \neq 0$), then for existence of $\alpha\gamma$ -centrality it has to be $\Delta > 0$, i.e. $\frac{1}{\alpha} < \lambda_2$ or $\frac{1}{\alpha} > \lambda_F$. But, by Lemma 3, $m \le \lambda_2$; thus it has to be $\frac{1}{\alpha} < m \le \lambda_2$. Thus, the $\alpha\gamma$ -centrality exists under $\alpha > \frac{1}{m}$.

If $1+\alpha(t_{12}-t_{22}) > 0, 1+\alpha(t_{21}-t_{11}) > 0,$ which meand $m < \frac{1}{\alpha},\ M < \frac{1}{\alpha}$, then for existence of $\alpha\gamma$ -centrality it has to be $\Delta < 0$, i.e. $\lambda_2 < \frac{1}{\alpha} < \lambda_F$. By Lemma 3, $m \le \lambda_2 \le M \le \lambda_F$. Hence, the $\alpha\gamma$ -centrality exists under $M < \frac{1}{\alpha} < \lambda_F$, i.e. under $\frac{1}{\lambda_F} < \alpha < \frac{1}{M}$.

3-4. Inequalities $t_{22} < t_{12}, t_{11} \le t_{21}$ imply $DetT < 0$; hence, $\lambda_2 < 0$.

If $\alpha > 0,\ \gamma < 0,$ then for existence of $\alpha\gamma$ -centrality it is needed $\Delta < 0$ (as is seen from (14), because $1+\alpha(t_{12}-t_{22}) > 0,\ 1+\alpha(t_{21}-t_{11}) > 0$). Then, by Lemma 3, needed is $\lambda_2 < \frac{1}{\alpha} < \lambda_F$. The inequality $\lambda_2 < \frac{1}{\alpha}$ is wittingly satisfied. Hence, the $\alpha\gamma$ -centrality exists under $\alpha > \frac{1}{\lambda_F}$.

If $\alpha < 0,\ \gamma > 0,$ then two cases are possible, under which, correspondingly, $\Delta < 0$ and $\Delta > 0$.

If $1+\alpha(t_{12}-t_{22}) < 0,\ 1+\alpha(t_{21}-t_{11}) < 0,$ which means $\frac{1}{\alpha} > M$, then for existence of $\alpha\gamma$ -centrality it has to be $\Delta < 0$, i.e. $\lambda_2 < \frac{1}{\alpha} < \lambda_F$. The inequality $\frac{1}{\alpha} < \lambda_F$ is wittingly satisfied. Let us consider inequalities $\lambda_2 < \frac{1}{\alpha}$, $\frac{1}{\alpha} > M$. By Lemma …, $\lambda_2 \le M$, i.e. the $\alpha\gamma$-centrality exists under $\alpha < \frac{1}{M}$.

If $1+\alpha(t_{12}-t_{22})>0$, $1+\alpha(t_{21}-t_{11})>0$, what means $\frac{1}{\alpha} < m$, that for existence of $\alpha\gamma$-centrality it has to be $\Delta > 0$, i.e., accounting for $\alpha < 0$, the inequality $\frac{1}{\alpha} < \lambda_2$ is checked. Let us consider the inequalities $\frac{1}{\alpha} < m$, $\frac{1}{\alpha} < \lambda_2$. By Lemma 3, $m \le \lambda_2 \le M \le \lambda_F$. Hence, $\alpha\gamma$-centrality exists under $\frac{1}{\alpha} < m.$, i.e. under $\frac{1}{m} < \alpha < 0.$

5. We have $t_{22} < t_{21}$, $t_{11} > t_{21}$.

First, let us consider the case of DetT > 0, then $\lambda_2 > 0.$ If $\alpha > 0,\ \gamma < 0,$ then:

$\Delta < 0,$ and hence, $\frac{1}{\lambda_F} < \alpha < \frac{1}{\lambda_2},$

$1+\alpha(t_{21}-t_{11})>0,$ and hence, $\alpha < \frac{1}{M}.$

By Lemma 3, $\frac{1}{\lambda_F} \le \frac{1}{M} \le \frac{1}{\lambda_2}$, hence, the condition of existence of $\alpha\gamma-$centrality is $\frac{1}{\lambda_F} < \alpha < \frac{1}{M}$.

If $\alpha < 0,\ \gamma > 0,$ then:

$\Delta > 0,$ and hence, $\alpha > \frac{1}{\lambda_F}$ or $\alpha < \frac{1}{\lambda_2}$; here active is only inequality $\alpha < 0$;

$1+\alpha(t_{12}-t_{22})>0,$ and hence, the $\alpha\gamma-$centrality exists under $\frac{1}{m} < \alpha < 0$.

In result, the condition of existence of $\alpha\gamma-$centrality is $\alpha \in (\frac{1}{m},0) \cup (\frac{1}{\lambda_F},\frac{1}{M}).$

Now, let DetT < 0, then $\lambda_2 < 0.$ If $\alpha > 0,\ \gamma < 0,$ then:

$\Delta < 0,$ and, hence, $\alpha > \dfrac{1}{\lambda_F},$

$1+\alpha(t_{21}-t_{11}) > 0,$ and, hence, $\alpha < \dfrac{1}{M}$.

By Lemma 3, $\dfrac{1}{\lambda_F} \le \dfrac{1}{M}$; hence, condition of existence of $\alpha\gamma-$centrality is $\dfrac{1}{\lambda_F} < \alpha < \dfrac{1}{M}$.

If $\alpha < 0, \gamma > 0,$ then:

$\Delta > 0,$ and hence, $\alpha > \dfrac{1}{\lambda_F}$ or $\alpha > \dfrac{1}{\lambda_2}$; active is inequality $\dfrac{1}{\lambda_2} < \alpha;$

$1+\alpha(t_{12}-t_{22}) > 0,$ and, hence, $\dfrac{1}{m} < \alpha.$,

By Lemma 3, $\dfrac{1}{\lambda_2} \le \dfrac{1}{m},$ hence, condition of existence of $\alpha\gamma-$centrality is $\dfrac{1}{m} < \alpha < 0.$

In result: $\alpha \in (\dfrac{1}{m},0) \cup (\dfrac{1}{\lambda_F},\dfrac{1}{M}).$

□

Example. A. Let us consider a network with type adjacency matrix $T = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. For this matrix, $m = t_{22} - t_{12} = -2,$ $M = t_{11} - t_{21} = -1,$ $\lambda_F = \sqrt{2}.,$ $\lambda_2 = -\sqrt{2}.$ The $\alpha\gamma-$centrality exists under

$$\alpha \in (-\infty,-1) \cup (-\frac{1}{2},0) \cup (\frac{1}{\sqrt{2}},+\infty).$$

B. Now, let new links appear among the subset of 1st type nodes. We will see that in result the constraint on the existence of $\alpha\gamma$- centrality becomes weaker. Let the new network have typology $T = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$. Then $t_{22} - t_{12} = -2 = m,$ $t_{11} - t_{21} = -1 = M,$

$\lambda_F = \dfrac{1+\sqrt{17}}{2},$ $\lambda_2 = \dfrac{1-\sqrt{17}}{2}$. The $\alpha-$centrality exists under

$$\alpha \in (-\infty,-1) \cup (-\frac{1}{2},0) \cup (\frac{2}{1+\sqrt{17}},+\infty).$$

The frontier of the right area has moved to the left: $\frac{2}{1+\sqrt{17}} < \frac{1}{\sqrt{2}}$.

The conditions of Theorem 7 have a transparent economic sense. For example, conditions $t_{12} \geq t_{22}$, $t_{21} \geq t_{11}$ mean that for each type the agent-outsider pays this type not less attention than the agent-insider does. This condition is rather common; for example, it is satisfied for 5 of 6 typologies with two types of nodes with degrees 3 and 2, shown in Figure 5 (the exclusion is the case d: typology $T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$).

Park, Barabasi, 2007 introduce, for the network with two kinds of agents (differing by an attribute), the following two characteristics of links of an agent with agents of the own and of the different kinds. We will provide definitions and formulas for calculation of these characteristics in our terms (types); m is the number of edges in the network.

Definition 2. Dyadicity of nodes of type i (i=1,2) is the ratio of the number of edges among the i-th type agents to their expected number under random uniform connection of nodes:

$$D_i = \frac{t_{ii} n_i n(n-1)}{2n_i(n_i-1)m} = t_{ii} \frac{n(n-1)}{2m(n_i-1)}, \quad i = 1,2.$$

. **Definition 3**. Heterophilicity is the ratio of the number of edges between agents of opposite types to their expected number under random uniform connection of nodes:

$$H = \frac{t_{12} n_1 n(n-1)}{2n_1 n_0 m} = t_{12} \frac{n(n-1)}{2mn_2} = t_{21} \frac{n(n-1)}{2mn_1}.$$

Conditions for the items 1, 3, 5 of the Theorem… can be written in terms of dyadicities and heterophilicity in the following way:

1) $\frac{D_i}{H} > \frac{n_i}{n_i - 1}$, $i = 1,2.$

(both types have relatively high dyadicity in comparison with heterophilicity);

3) $\frac{D_i}{H} < \frac{n_i}{n_i - 1}$, $i = 1,2.$

(both types have relatively high heterophilicity in comparison with dyadicity);

5) $\frac{D_2}{H} < \frac{n_2}{n_2 - 1}, \ \frac{D_1}{H} > \frac{n_1}{n_1 - 1}$

(one of the types has a relatively high heterophilicity in comparison with dyadicity, and the other type has relatively high diadicity in comparison with heterophilicity).

Thus, Theorem 7 may be interpreted in the following way. If two types are look like, in the sense that for both of them dyadicity is relatively high (or relatively low) in comparison with heterophilicity, then the $\alpha\gamma$-centrality exists under α arbitrarily high in absolute value. But if the types are not look like to each other, in the sense that for one of them dyadicity is relatively high in comparison with heterophilicity, and for another one dyadicity is relatively small in comparison with heterophilicity, then the $\alpha\gamma$-centrality exists only under some limited by absolute value α.

It is possible to make more precise the result of Theorem 5 concerning the coincidence of the orders generated by degrees and by the $\alpha\gamma$ – centrality. We continue to assume

$$m = t_{22} - t_{12} < M = t_{11} - t_{21}.$$

Corollary. *1) $\alpha\gamma$ – centrality exists and generates the same order as degrees in the following cases:*

1. $m > 0, \ \alpha \in (-\infty, 0) \cup (\frac{1}{m}, +\infty)$.

2. $m = 0, \ \alpha \in (-\infty, 0)$.

3. $m < 0, \ \alpha \in (\frac{1}{m}, 0)$.

2) $\alpha\gamma$ – centrality exists and generates the order opposite to degrees in the following cases:

1. $m \geq 0, \ \alpha \in (\frac{1}{\lambda_F}, \frac{1}{M})$.

2. $M \leq 0, \ \alpha \in (-\infty, \frac{1}{M}) \cup (\frac{1}{\lambda_F}, +\infty)$.

3. $m < 0, \ M > 0, \ \alpha \in (\frac{1}{\lambda_F}, \frac{1}{M})$.

3) *$\alpha\gamma$ – centrality does not exist in the following cases:*

1. $m > 0,\ \alpha \in (0, \frac{1}{\lambda_F}) \cup (\frac{1}{M}, \frac{1}{m})$.

2. $m = 0,\ \alpha \in (0, \frac{1}{\lambda_F}) \cup (\frac{1}{M}, +\infty)$.

3. $M < 0,\ \alpha \in (\frac{1}{M}, \frac{1}{m}) \cup (0, \frac{1}{\lambda_F})$.

4. $M = 0,\ \alpha \in (-\infty, \frac{1}{m}) \cup (0, \frac{1}{\lambda_F})$.

5. $m < 0,\ M > 0,\ \alpha \in (-\infty, \frac{1}{m}) \cup (0, \frac{1}{\lambda_F}) \cup (\frac{1}{M}, +\infty)$.

Proof. The statement directly follows from the results of Theorem 5 and Theorem 7 taking into account the identification for the cases $\Delta > 0$ и $\Delta < 0$, which was obtained in the proof of Theorem 7.

9. Conclusion

Models of network economics and network games show that agents' behaviors in equilibrium are defined by their positions in the network, which are described by one or other centrality measure. However, various centrality measures prove important in different models. We consider two actual questions of analysis of centrality measures in networks. First, what is the interrelation between common centrality measures and their relation with other structural characteristics of networks? Second, in which cases different centrality measures generate the same order of nodes in network?

We find a class of centrality measures (which includes degree, Katz-Bonacich centrality, eigenvector centrality, diffusion centrality, alpha-beta centrality, and alpha-gamma centrality) which characterize not just separate nodes but types of nodes selected by a rather universal structural characteristic – a network typology. To each network (undirected graph) a 'type adjacency matrix' T corresponds which shows for each type numbers of neighbors of different types. Two networks are said to have the same typology if they have the same type adjacency matrix. Knowledge of matrix T (which may have much lower dimensionality than adjacency matrix A) is sufficient to calculate the centrality measures of the above-mentioned class. We prove that in any class of networks with same fixed typology, if i an j are nodes of the same type (may be even belonging different networks of this class), then c(i)=c(j), where c is any of the centrality measures of the above-mentioned class. This implies that networks of different size but with the same typology have common properties in economic models; in particular, game equilibria may be transplanted among networks of the same typology.

The second basic result of the paper is that the nearest generalization of the familiar class of regular networks, the class of networks with two types of nodes, possesses a remarkable property: the centrality measures of the above-mentioned class provide the same order on the set of nodes. Recently Bloch et al., 2017 found another class of centrality measures and another class of networks (so called regular monotonous hierarchies), which possess a similar property. The intersection of the classes of networks found in Bloch et al., 2017 and in our paper is the class of star networks.

The results lead to new natural questions. One of them, is what are the common characteristics of the centrality measures belonging to each of the classes found in Bloch et al., 2017 and in the present paper? Another question is what is the relation between the two classes of networks: the regular monotonous hierarchies and the two-type-of-nodes networks?

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